Rigorous scaling law for the heat current in disordered harmonic chain

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Abstract

We study the energy current in a model of heat conduction, first considered in detail by Casher and Lebowitz. The model consists of a one-dimensional disordered harmonic chain of n i.i.d. random masses, connected to their nearest neighbors via identical springs, and coupled at the boundaries to Langevin heat baths, with respective temperatures T_1 and T_n. Let $\mathsf{E}J_n$ be the steady-state energy current across the chain, averaged over the masses. We prove that $\mathsf{E}J_n \sim (T_1 - T_n)n^{-3/2}$ in the limit $n \to \infty$, as has been conjectured by various authors over the time. The proof relies on a new explicit representation for the elements of the product of associated transfer matrices.

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1 Introduction

In a bulk of material, Fourier's law is said to hold if the flux of energy J is proportional to the gradient of temperature, i.e.,

$$J = -\kappa \nabla T, \tag{1.1}$$

where κ is called the conductivity of the material. This phenomenological law has been widely verified in practice. Nevertheless, the mathematical understanding of thermal conductivity starting from a microscopic model is still a challenging question [4] [9] (see also [14] for a historical perspective).

Since the work of Peierls [18][19], it has been understood that anharmonic interactions between atoms should play a crucial role in the derivation of Fourier's law for perfect crystals. It has been known for a long time that the conductivity of perfect harmonic crystals is infinite. Indeed, in this case, phonons travel ballistically without any interaction. This yields a wave like transport of energy across the system, which is qualitatively different than the diffusion predicted by the Fourier law (1.1). For example, in [21], it is shown that the energy current in a one-dimensional perfect harmonic crystal, connected at each end to heat baths, is proportional to the difference of temperature between these baths, and not to the temperature gradient.

In addition to the non-linear interactions, also the presence of impurities causes scattering of phonons and may therefore strongly affect the thermal conductivity of the crystal. Thus, while avoiding formidable technical difficulties associated to anharmonic potentials, by studying disordered harmonic systems one can learn about the role of disorder in the heat conduction. Moreover, many problems arising with harmonic systems can be stated in terms of random matrix theory, or can be reinterpreted in the context of disordered quantum systems.

Indeed, in [8] Dhar considered a one-dimensional harmonic chain of n oscillators connected to their nearest neighbors via identical springs and coupled at the boundaries to the rather general heat baths parametrized by a function $\mu : \mathbb{R} \to \mathbb{C}$ and the temperatures T_1 and T_n of the left and right baths, respectively. Dhar expressed the steady state heat current $J_n^{(\mu)}$ as the integral over oscillation frequency w of the modes:

$$J_n^{(\mu)} = (T_1 - T_n) \int_{\mathbb{R}} \left| v_{\mu,n}^{\mathrm{T}}(w) A_n(w) \cdots A_1(w) v_{\mu,1}(w) \right|^{-2} \mathrm{d}w \,. \tag{1.2}$$

Here $A_k(w) \in \mathbb{R}^{2\times 2}$ is the random transfer matrix corresponding the mass of the k^{th} oscillator, while $v_{\mu,1}(w)$ and $v_{\mu,n}(w)$ are $\mathbb{C}^2$-vectors determined by the bath function μ and the masses of the left and the right most oscillators, respectively. Standard multiplicative ergodic theory [2] tells that asymptotically the norm of $Q_n(w) := A_n(w) \cdots A_1(w)$ grows almost surely like $\mathrm{e}^{\gamma(w)n}$ where the non-random function $\gamma(w) \geq 0$ is the associated Lyapunov exponent. In the context of heat conduction this corresponds the localization of the eigenmodes of one-dimensional chains while in disordered quantum systems one speaks about the one-dimensional Anderson localization [1].

However, in the absence of an external potential (pinning), the Lyapunov exponent scales like w^2, when w approaches zero, and this makes the scaling behavior of (1.2) non-trivial as well as highly dependent on the properties of the bath. Indeed, only those modes for which the localization length $1/\gamma(w)$ is of equal or higher order than the length of the chain, n, do have a non-exponentially vanishing contribution in (1.2). Thus the heat conductance of the chain depends crucially on how the bath vectors $v_{\mu,1}(w), v_{\mu,n}(w)$ weight the critical frequency range $w^2 n \lesssim 1$. In other words, explaining the scaling of the heat current in disordered harmonic chains reduces to understanding the limiting behavior of the matrix product $Q_n(w)$ when $w \leq n^{-1/2+\epsilon}$ for some $\epsilon > 0$.

The evolution of $n \mapsto Q_n(w)$ reaches stationarity only when $w^2 n \sim 1$ while the components of $Q_n(w)$ oscillate in the scale $wn \sim 1$ with a typical amplitude of $w^{-1}\mathrm{e}^{\gamma_0 w^2 n}$ as observed numerically in [8]. Thus the challenge when working in this small frequencies regime is that the analysis does fall back neither to classical asymptotic estimates for large n, nor to the estimate of the Lyapunov exponent for small w.

Of course, the difficulty of this analysis depends also on the exact form of the vectors $u_{\mu,k}$ in (1.2), i.e., on the choice of the heat baths. Besides some rather recent developments, most of the studies so far have concentrated on two particular models. In the first model, introduced by Rubin and Greer [22], the heat baths themselves are semi-infinite ordered harmonic chains distributed according to Gibbs equilibrium measures of temperatures T_1 and T_n, respectively. Rubin and Greer were able to show that $\mathsf{E}J_n^{\mathrm{RG}} \gtrsim n^{-1/2}$ with $\mathsf{E}[\,\bullet\,]$ denoting the expectation over the masses. Later Verheggen [23] proved that $\mathsf{E}J_n^{\mathrm{RG}} \sim n^{-1/2}$.

In the second model the heat baths are modeled by adding stochastic Ornstein-Uhlenbeck terms to the Hamiltonian equations of the chain (see (1.4) below). This model, first analyzed by Casher and Lebowitz [5] in the context of heat conduction, was conjectured by Visscher (see ref. 9 in [5]) to satisfy $\mathsf{E}J_n^{\mathrm{CL}} \sim n^{-3/2}$. Moreover, already in [5] it was argued that $\mathsf{E}J_n^{\mathrm{CL}} \gtrsim n^{-3/2}$. However, the line of reasoning there contains an error which invalidates this lower bound (see Section 6), and therefore no rigorous upper nor lower bounds have been published for $\mathsf{E}J_n^{\mathrm{CL}}$ until now.

1.1 Casher-Lebowitz model and results

The Hamiltonian of the isolated one-dimensional disordered chain is

$$H(q_1,\dots q_n,p_1,\dots p_n) \;=\; \sum_{k=1}^{n}\frac{p_k^2}{2m_k} \;+\; \frac{1}{2}\sum_{k=0}^{n}(q_{k+1}-q_k)^2\,, \tag{1.3}$$

where $q_k \in \mathbb{R}$ is the displacement of the k^{th} mass m_k from its equilibrium position and p_k is the associated momentum. We consider fixed boundaries, i.e., $q_0 = q_{n+1} = 0$.

The usual Hamilton's equations are modified at the endpoints in order to include an interaction with heat baths. In the Casher-Lebowitz model, this interaction consists of adding white noise and a viscous friction terms to the Hamiltonian equations of p_1 and p_n: Suppose $\lambda > 0$ is the coefficient of viscosity, let $T_1 \geq T_n > 0$ be the respective temperatures of the reservoirs, and let W_1, W_n be two independent Brownian motions. The equations of motion for the Casher-Lebowitz chain then take the form of the stochastic differential equation

$$\begin{aligned}\mathrm{d}q_k \;&=\; \frac{\partial H}{\partial p_k}\mathrm{d}t\,,\\ \mathrm{d}p_k \;&=\; -\frac{\partial H}{\partial q_k}\mathrm{d}t \;+\; (\delta_{k,1}+\delta_{k,n})(-\lambda p_k\mathrm{d}t + \sqrt{2\lambda T_k m_k}\,\mathrm{d}W_k)\,,\end{aligned} \tag{1.4}$$

with $1 \leq k \leq n$. If $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$, then, as far as the scaling behavior goes, the choice (1.4) of heat baths corresponds (see [5], and (2.5) below) to setting $v_{\mu,1}(w) = |w|^{-1/2}e_1 + \mathrm{i}|w|^{1/2}e_2$ and $v_{\mu,n}(w) = |w|^{-1/2}e_1 - \mathrm{i}|w|^{1/2}e_2$ in (1.2). The resulting current, denoted by $J_n^{\mathrm{CL}}(m_1,\dots,m_n)$, is then by definition the average rate at which energy is carried from the left to the right heat bath over the stationary measure of (1.4) for fixed masses m_k.

Now, suppose that the masses are random variables M_k. Our main result is the following strict scaling relation for the mass averaged stationary current.

THEOREM 1.1. *Assume that the masses $(M_k : k \in \mathbb{N})$ are independent and identically distributed. Suppose that the common probability distribution of the masses M_k admits a density, compactly supported on $]0,\infty[$, continuously differentiable inside its support, with an uniformly bounded derivative. Denote by $\mathsf{E}[\,\bullet\,]$ the expectation over the masses. Then there exist $K, K' > 0$ such that the heat current J_n^{CL} satisfies the relation*

$$K\,\frac{T_1 - T_n}{n^{3/2}} \;\leq\; \mathsf{E}\big[J_n^{\mathrm{CL}}(M_1,\dots M_n)\big] \;\leq\; K'\,\frac{T_1 - T_n}{n^{3/2}}\,. \tag{1.5}$$

The proof is based on a new representation of the matrix $Q_n(w)$ in terms of a discrete time Markov chain on a circle. Based on this representation we obtain a good control of the joint behavior of the matrix elements of $Q_n(w)$ for the most important regime $w \le n^{-1/2+\epsilon}$ where $\epsilon > 0$ is small. Moreover, together with O'Connor's decay estimates [17] for high frequencies we have a good control of the exponential decay of $\|Q_n(w)\|$ whenever $w \ge n^{-1/2+\epsilon}$. Therefore, the possibility of generalizing Theorem 1.1 to a quite large class of heat baths seems possible by extending our analysis. Indeed, in Subsection 6.3 we sketch how one can derive the scaling behavior of the stationary heat current for Dhar's modified version of the Casher-Lebowitz model as well as to prove the analogue of Theorem 1.1 for the Rubin-Greer model.

The organization of the paper is as follows. In Section 2, we present the practical expression for the current J_n^{CL}, after first introducing some conventions and notation to be used in the rest of the paper. In the end of Section 2 our strategy to obtain Theorem 1.1 is outlined. Sections 3 to 5 contain the three main technical results needed for the proof. The actual proof of Theorem 1.1 is then presented in Section 6.

2 Conventions and outline of paper

For the rest of this manuscript we are going to *assume that the conditions of Theorem 1.1 hold.* In particular, this means that the zero mean random variables

$$B_k \;:=\; \frac{M_k - \mathsf{E} M_k}{\mathsf{E} M_k}\,, \tag{2.1}$$

are i.i.d., have a (Lebesgue) probability density τ that satisfies $\mathrm{supp}(\tau) \subset [b_-, b_+]$, and $\tau \in \mathrm{C}^1([b_-, b_+])$, for some constants $-1 < b_- < b_+ < \infty$. Here $\mathrm{C}^k([a,b])$ denotes a continuous function $f : [a,b] \to \mathbb{R}$ such that $\frac{\mathrm{d}^j f}{\mathrm{d}x^j}$ exist for $j \le k$, and that these derivatives are bounded and continuous on $]a,b[$. The transfer matrices appearing in (1.2) are related to B_k:

$$A_k \;\equiv\; A_k(w) \;=\; \begin{bmatrix} 2 - \pi^2 w^2 (1 + B_k) & -1 \\ 1 & 0 \end{bmatrix}, \tag{2.2}$$

where the frequency variable w is related to the frequency variable ω in [5] by $\omega = \pi^{-1}(\mathsf{E} M_k)^{1/2} w$. As already pointed out in the introduction, O'Connor has shown (see Theorem 6 and its proof in [17]) that for any reasonable heat baths the frequencies above any fixed $w_0 > 0$ have exponentially small contribution to the total current (1.2) as n grows. Therefore, one may *consider an arbitrary small but fixed interval* $]0, w_0]$ *of frequencies* w *in order to prove Theorem 1.1.*

We write $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$ and $\mathbb{R}_+ = \,]0,\infty[$ and $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with $\infty = \pm\infty$. Additionally, following conventions are used frequently.

Probability: Since all the randomness of the stationary state current J_n^{CL} originates from the random masses we define the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ as the semi-infinite countable product of spaces $([b_-, b_+], \mathcal{B}[b_-, b_+], \tau(b)\mathrm{d}b)$. Here $\mathcal{B}(S)$ denotes the Borel σ-algebra of the topological space S. The filtration generated by the sequence $B \equiv (B_k : k \in \mathbb{N})$ is denoted by $\mathbb{F} = (\mathcal{F}_k : k \in \mathbb{N})$, $\mathcal{F}_k = \sigma(B_j : 1 \le j \le k) \subset \mathcal{F}$. As a convention, the names of new random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ will be generally written in capital letters. A discrete time stochastic process $(Z_n : n \in \mathbb{K})$ is denoted by $Z \equiv (Z_n)$ when index set $\mathbb{K}$ is known or not relevant. Finally, we write $\Delta Z_n = Z_n - Z_{n-1}$.

Constants and scaling: Because we are interested only in the scaling relations many expressions can be made more manageable by using the following conventions. First, we use letters $C, C', C_1, C_2, \ldots$ to denote strictly positive finite constants, whose value may vary from place to place. Except otherwise stated, these values depend only on $\tau, \lambda, T_1 - T_n$ and w_0, but never on

w or n. Secondly, suppose f, g, h are functions, we write $f \lesssim g$, or equivalently, $g \gtrsim f$ provided $f \leq C\, g$ pointwise, i.e., $f(x, y) \leq C g(y, z)$ for all possible arguments x, y, z. If $f \lesssim g$ and $f \gtrsim g$ then we write $f \sim g$. Moreover, the expression $f = g + \mathcal{O}(h)$, where f, g, h means $|f - g| \leq C|h|$.

Periodicity: In the following we are going to deal with functions that are defined and/or take values on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The following conventions are practical on such occasions. When $x \in \mathbb{R}$ write $|x|_{\mathbb{T}} = \min(x - \lfloor x \rfloor, \lceil x \rceil - x)$ where $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the largest (smallest) integer smaller (larger) than x. We identify 1-periodic functions on $\mathbb{R}$ with functions on $\mathbb{T}$. Similarly, a function $g : \mathbb{R} \to \mathbb{R}$ of the form $g(x) = x + f(x)$, where f is 1-periodic, is identified with a function from $\mathbb{T}$ to itself.

2.1 Heat current in terms of matrix elements

Let $v = [v_0 \ v_{-1}]^{\mathrm{T}} \in \mathbb{C}^2$, and denote by $D(v) \equiv (D_n(v) : n \in \mathbb{N})$ the discrete time stochastic process that solves for $n \in \mathbb{N}$:

$$\begin{aligned} D_n(v) &= (1 - \pi^2 w^2 (1 + B_n)) D_{n-1}(v) \; - \; D_{n-2}(v) \\ D_0(v) &= v_0 \,, \\ D_{-1}(v) &= v_{-1} \,. \end{aligned} \tag{2.3}$$

By definition one then has for $n \in \mathbb{N}$

$$Q_n \;=\; A_n A_{n-1} \cdots A_1 \;=\; \begin{bmatrix} D_n(e_1) & D_n(e_2) \\ D_{n-1}(e_1) & D_{n-1}(e_2) \end{bmatrix} , \tag{2.4}$$

where A_k is the transfer matrix (2.2) and $e_1 = [1 \ 0]^{\mathrm{T}}$ and $e_2 = [0 \ 1]^{\mathrm{T}}$. As a remark it is worth noting that in the derivation of the stationary heat current one actually starts with (2.3) where $D_n(e_k)$ are certain real valued (sub-)determinants of a semi-infinite matrix and then expresses the final formula conveniently in terms of the product (2.4).

Now, in [5] it was proven that Casher-Lebowitz model corresponds to setting the bath vectors $v_{\mu,1}$ and $v_{\mu,n}$ in the general expression (1.2) of $J_n^{(\mu)}$ equal to

$$v_{\mathrm{CL},1}(w) \;=\; \begin{bmatrix} (\alpha M_1 |w|)^{-1/2} \\ +\mathrm{i}(\alpha M_1 |w|)^{1/2} \end{bmatrix} \quad \text{and} \quad v_{\mathrm{CL},n}(w) \;=\; \begin{bmatrix} (\alpha M_n |w|)^{-1/2} \\ -\mathrm{i}(\alpha M_n |w|)^{1/2} \end{bmatrix} . \tag{2.5}$$

Here the constant $\alpha > 0$ depends on the units of the frequency variable w, etc. Since the masses have a compact support, $[m_-, m_+] \subset]0, \infty[$ and the bath vectors are symmetric in w, one has

$$J_n^{\mathrm{CL}} \;\sim\; (T_1 - T_n) \int_{\mathbb{R}} |v_{\mathrm{CL},n}^{\mathrm{T}}(w) Q_n(w) v_{\mathrm{CL},1}(w)|^{-2} \mathrm{d}w \;\sim\; \int_0^\infty j_n(w) \mathrm{d}w \;=:\; J_n \,, \tag{2.6}$$

where $j_n(w) := |v_n^{\mathrm{T}}(w) Q_n(w) v_1(w)|^{-2}$, with $v_1(w) = w^{-1/2} e_1 + \mathrm{i} w^{1/2} e_2$ and $v_n(w) = w^{-1/2} e_1 - \mathrm{i} w^{1/2} e_2$. By using $D_n(e_1) D_{n-1}(e_2) - D_{n-1}(e_1) D_n(e_2) = \det(A_n \cdots A_1) = 1^n = 1$ to get rid of the mixed terms of $D_n(e_k) \equiv D_n(e_k; w)$ one obtains:

$$j_n(w) \;=\; \big\{ 1 + w^{-2} D_n(e_1)^2 + D_{n-1}(e_1)^2 + D_n(e_2)^2 + w^2 D_{n-1}(e_2)^2 \,\big\}^{-2} . \tag{2.7}$$

This is the form we are going to use for the proof of Theorem 1.1.

2.2 Outline of the proof

It follows from (2.6) and (2.7) that the scaling bounds of $\mathsf{E}(J_n^{\mathrm{CL}}) \sim \mathsf{E}(J_n)$ rely on the good understanding of the processes $D(v)$ defined in (2.3). Thus, the first natural step towards the proof of the theorem is the derivation of an easier representation for $D_n(v)$. This is the purpose of Section 3 where one constructs (Proposition 3.5 and Corollary 3.6) the representations:

$$D_n(e_1) \;\sim\; w^{-1}\Gamma_n^{\vartheta}\cdot \sin\pi X_n^{\vartheta}\,, \qquad \text{and} \qquad D_n(e_2) \;\sim\; w^{-1}\Gamma_n^{0}\cdot \sin\pi X_n^{0}\,. \tag{2.8}$$

Here $\vartheta = w + \mathcal{O}(w^3)$ is a constant, the phases $(X_n^x : n \in \mathbb{N}_0)$ form a Markov process on $\mathbb{T}$

$$X_n^x \;=\; X_{n-1}^x + w \,+\, w\phi(X_{n-1}^x)B_n \,+\, \mathcal{O}(w^2) \qquad \text{with} \qquad X_0^x = x\,, \tag{2.9}$$

and the amplitude $\Gamma_n^x \in]0,\infty[$ is an exponential functional of $(x, B_k : 1 \le k \le n)$:

$$\Gamma_n^x \;=\; \mathrm{e}^{w\sum_{k=1}^n s(X_{k-1}^x)B_k \,+\, w^2\sum_{k=1}^n r(X_{k-1}^x)B_k^2 \,+\, \mathcal{O}(w^3 n)}\,. \tag{2.10}$$

The smooth functions $\phi, s, r : \mathbb{T}\to\mathbb{R}$ are explicitly known. The process $X \equiv X^x$ is specified precisely in Definition 3.3 and Lemma 3.2, and its most important qualitative properties are listed in Corollary 3.4. The main advantage of the representation (2.8) is that, unlike the recursion relations (2.3) of $D(v)$, it allows us to treat both the scaled noise wB_n and the initial values e_2 of $D_n(e_2)$ as small perturbations around 0 and e_1, respectively.

Based on the representation (2.8), let us now carry out heuristic computations which form the outline for the actual proof of $\mathsf{E}(J_n) \sim n^{-3/2}$. Along these calculations we will point out the properties of X_n^x and Γ_n^x which must be proven to make these calculations rigorous. We start with the upper bound. By Theorem 6 of [17] we may restrict the integration domain of (2.6) into $[0, w_0]$. Dropping positive terms from the denominator in (2.7) then yields

$$\mathsf{E}J_n^{\mathrm{CL}} \;\sim\; \mathsf{E}J_n \;=\; \mathsf{E}\int_0^\infty j_n(w)\mathrm{d}w \;\le\; \int_0^{w_0} \mathsf{E}\left\{\frac{1}{1+w^{-2}D_n(e_1;w)^2}\right\}\mathrm{d}w \tag{2.11a}$$

$$\;=\; \int_0^{w_0} \mathsf{E}\left\{\int_{\mathbb{T}} \frac{1}{1+(w^{-2}\Gamma_n \sin x)^2}\,\mathsf{P}(X_n\in \mathrm{d}x|\Gamma_n)\right\}\mathrm{d}w\,. \tag{2.11b}$$

Now comes the first crucial step. By standard martingale central limit theorems [13] one expects that X_n, if properly centered, scaled, and considered as a process on $\mathbb{R}$, should converge to a Gaussian with unit variance. Unfortunately, such weak convergence results do not suffice since we need to deal with very unlikely events. Indeed, from (2.11b) one sees that the crucial contribution of the terms inside the curly brackets comes when $|X_n| \lesssim w^2/\Gamma_n$. The probability of this to happen is typically very small, e.g., of order n^{-1} when $w^2 n \sim 1$. Moreover, we would also like to be able to consider X_n and Γ_n effectively independent in (2.11b). In other words, we would like to have:

(a) Pointwise bound: $\chi_{B(wn, Cw\sqrt{n})}(x)\cdot \frac{\mathrm{d}x}{\min(1,w\sqrt{n})} \;\lesssim\; \mathsf{P}(X_n\in\mathrm{d}x) \;\lesssim\; \frac{\mathrm{d}x}{\min(1,w\sqrt{n})}$, $x\in\mathbb{T}$;

(b) Independence: $\mathsf{P}(X_n\in\mathrm{d}x|\Gamma_n) \;\sim\; \mathsf{P}(X_n\in\mathrm{d}x)$, $x\in\mathbb{T}$.

The purpose of Section 5 is to prove Proposition 5.1 which together with the bounds in Subsection 6.2 implies that as far as (2.11b) goes one may think that both (a) and (b) hold literally. So by using (a-b) and then parametrizing $\mathbb{T}$ with $[-1/2, 1/2]$ in (2.11b) one gets

$$\begin{aligned}
\mathsf{E}(J_n) \;&\lesssim\; \int_0^{w_0}\mathsf{E}\left\{\int_{-1/2}^{1/2}\frac{1}{1+(w^{-2}\Gamma_n x)^2}\cdot\frac{\mathrm{d}x}{\min(1,w\sqrt{n})}\right\}\mathrm{d}w \\
&\lesssim\; \int_0^{w_0}\frac{1}{\min(1,w\sqrt{n})}\mathsf{E}\left\{\frac{\arctan(w^{-2}\Gamma_n)}{w^{-2}\Gamma_n}\right\}\mathrm{d}w \\
&\lesssim\; \int_0^{n^{-1/2}}\frac{w}{\sqrt{n}}\,\mathsf{E}\{1/\Gamma_n(w)\}\,\mathrm{d}w \,+\, \int_{n^{-1/2}}^{w_0}\mathsf{E}\{1/\Gamma_n(w)\}\,\mathrm{d}w\,.
\end{aligned} \tag{2.12}$$

Here we have used the upper bound in (a), approximated $\sin z \sim z$ and then performed a change of variables $x \mapsto w^{-2}\Gamma_n x$. To get the last line we have approximated $\arctan r \lesssim 1$, for $r \in \mathbb{R}_+$.

In Section 4 we bound the only unknown term in (2.12) by showing that there exists a constant $\alpha > 0$ such that

$$\mathsf{E}\{1/\Gamma_n(w)\} \;\lesssim\; \mathrm{e}^{-\alpha w^2 n}\,, \quad \text{when} \quad 0 < w \leq w_0\,. \tag{2.13}$$

The sum over r-terms in (2.10) is then shown to produce an exponent $\mathrm{e}^{-\gamma(w)n}$ where the constant $\gamma(w) \sim w^2$ is the Lyapunov exponent associated to the transfer matrices A_k in (2.2) with explicit value given in (4.2). The challenge in Section 4 is to bound the large deviations of the first sum in (2.10) so much that (2.13) still holds for some $\alpha > 0$. By applying the bound (2.13) in (2.12), yields the upper bound for the total current:

$$\mathsf{E}(J_n) \;\lesssim\; \int_0^{n^{-1/2}} \frac{w}{\sqrt{n}} \cdot 1\,\mathrm{d}w \,+\, \int_{n^{-1/2}}^{w_0} w^2 \mathrm{e}^{-\gamma w^2 n} \mathrm{d}w \;\sim\; n^{-3/2}\,.$$

To prove the lower bound, it suffices to show that for $w \in I := [(2n)^{-1/2}, n^{-1/2}]$ one has $\mathsf{P}\big(j_n(w) \geq Cw^2\big) \gtrsim 1$. Indeed, if this bound is verified then

$$\mathsf{E}(J_n) \gtrsim \int_I \mathsf{E} j_n(w)\mathrm{d}w \;\geq\; n^{-1/2} \cdot C(n^{-1/2})^2 \cdot \mathsf{P}\big(j_n(w) \geq Cw^2\big) \;\sim\; n^{-3/2}\,.$$

Just like with the upper bound the main contribution of $\mathsf{E} j_n(w)$ comes from the unlikely events, e.g., when $|X_n| \lesssim w^2$. For this reason one needs again the pointwise bounds (a) and (b). However, unlike in (2.11a) the lower bound depends in a non-trivial way also on $D_n(e_2)$ since by (2.7) one has

$$\mathsf{P}\big(j_n(w) \geq C_1 w^2\big) \;\sim\; \mathsf{P}\big(|D_n(e_1; w)| \leq w^2, |D_n(e_2; w)| \leq w\big) \tag{2.14}$$

Thus, to prove the lower bound one has to be able to analyze the joint behavior of the matrix elements $(D_n(e_1), D_n(e_2))$, or equivalently, $(X_n^\vartheta, X_n^0, \Gamma_n^\vartheta, \Gamma_n^0)$. These dependencies are first addressed in Subsection 3.2 by deriving martingale exponent representations for both $X_n^\vartheta - X_n^0$ and $\Gamma_n^0/\Gamma_n^\vartheta$. In Subsection 6.1 these representations are used to extract (Lemma 6.1) the typical joint behavior of the processes $D(e_k)$, $k = 1, 2$. Based on this typical behavior one is then able to construct the final bound for the right side of (2.14).

3 Representation of matrix elements

The purpose of this section is to derive the representation (2.8) of processes $D(v)$, $v \in \mathbb{R}^2$, (Proposition 3.5 and Corollary 3.6) in terms of the Markov process (X_n) on the unit circle $\mathbb{T}$. The first step of this derivation is to use the Möbius transformation, associated to the average of the transfer matrix $\mathsf{E}(A_n)$, to construct w-depended change-of-coordinates g which maps the evolution of the quotients $\xi_n = D_n/D_{n-1}$ bijectively from $\bar{\mathbb{R}}$ to $\mathbb{T}$. It turns out that in these new coordinates $x = g^{-1}(\xi)$ the noise, wB_n, can be considered as a small perturbation around the zero noise evolution, which in turn is reduced to the simple shift $x \mapsto x + \vartheta$. This is unlike in the original coordinates $\xi \in \bar{\mathbb{R}}$ where the effect of noise is typically of order $\mathcal{O}(1)$ regardless how small w is. The Markov process (X_n) is now defined by $X_n := g^{-1}(D_n/D_{n-1})$ while the representation for the matrix elements is obtained by first writing $D_n = g(X_n) \cdots g(X_1) \cdot D_0$ and then using the explicit knowledge of g for expanding the resulting expression w.r.t. the small disorder $(wB_n : n \in \mathbb{N})$.

The representation (2.8) is new. Besides having the benefits already mentioned before, it also has the nice property of reducing in the zero noise case to the explicit expression $D_{1,n} \equiv D_n =$

$\frac{\sin \pi\vartheta(n+1)}{\pi\vartheta}$ which was already discovered by Casher and Lebowitz (consider 1-periodic chain in equation (3.5) in [5]). The change-of-coordinates g, on the other hand, is not really new as it was already discovered in a slightly different form by Matsuda and Ishii [15]. However, since our method of deriving g is different than in [15] we have decided to include it here for the convenience of the reader.

In a more general context, our representation (2.8) is similar to some standard decomposition of products on Markov chains. Indeed, since $D_n = \xi_n \cdots \xi_1 D_0$ with $\xi_k = D_n/D_{n-1}$, and since the transfer operator of the chain (ξ_n) admits a spectral gap [17], a general argument [12] allows us to write the decomposition $|D_n| = \mathrm{e}^{\gamma n + M_n} u(\xi_n)$, where γ is a Lyapunov exponent, (M_n) is a martingale, and u is a function on $\mathbb{R}$. Although, one is not in general able to determine M_n and u, it turns out that, in the special case of random matrices, Raugi [20] has been able to compute them explicitly, up to the knowledge of the invariant measure of the chain (ξ_n). Still, the derivation of our formula (2.8) is much more straightforward than the use of Raugi's formula.

3.1 Expansion around zero noise evolution

Let us associate a Möbius transformation $\mathcal{M}_A : \mathbb{C} \to \mathbb{C}$ to a 2×2 to a square matrix A by setting

$$\mathcal{M}_A(z) := \frac{az+b}{cz+d} \qquad \text{for} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

The association $A \mapsto \mathcal{M}_A$ preserves the matrix multiplication

$$\mathcal{M}_A \circ \mathcal{M}_B = \mathcal{M}_{AB} , \qquad (A, B \in \mathbb{C}^{2\times 2}) \tag{3.1}$$

so that $(\mathcal{M}_A)^{-1} = \mathcal{M}_{A^{-1}}$ whenever either side of the equality exists.

By writing $D_n \equiv D_n(v)$, $v = [v_0 \; v_{-1}]^{\mathrm{T}} \in \mathbb{C}^2$, and using (2.3) one sees that the ratios

$$\xi_n := \frac{D_n}{D_{n-1}} , \tag{3.2}$$

form a Markov process $\xi \equiv (\xi_n : n \in \mathbb{N}_0)$ which satisfies a simple recursion relation:

$$\xi_n = \mathcal{M}_{A_n}(\xi_{n-1}) \qquad (n \in \mathbb{N}) \tag{3.3a}$$

$$\xi_0 = \frac{v_0}{v_{-1}} . \tag{3.3b}$$

Here the random matrices A_n depend on B_n through the relation (2.2). Since $\mathcal{M}_{A_n}(\pm\infty) = 2 - \pi^2 w^2 (1 + B_n)$ we identify $\pm\infty = \infty$. By using (3.2) and (3.3) we get

$$D_n = \xi_n \xi_{n-1} \cdots \xi_1 D_0 , \tag{3.4}$$

provided no $\xi_k \in \{0, \infty\}$. Recall that $\bar{\mathbb{R}}$ denotes $\mathbb{R} \cap \{\infty\}$. In the following we shall consider (3.3) on $\bar{\mathbb{R}}$ instead on $\mathbb{C} \cap \{\infty\}$.

LEMMA 3.1. *There exists a coordinate transformation $g : \mathbb{T} \to \bar{\mathbb{R}}$ such that*

$$(g^{-1} \circ \mathcal{M}_{\mathsf{E}(A_k)} \circ g)(x) = x + \vartheta \qquad (x \in \mathbb{T}) , \tag{3.5}$$

where A_k is the random matrix (2.2), and the constant shift is given by

$$\vartheta \equiv \vartheta(w) = \frac{1}{\pi} \arccos \left[1 - \frac{\pi^2 w^2}{2} \right] = w + \mathcal{O}(w^3) . \tag{3.6}$$

The function g and and its inverse g^{-1} are given by

$$g(x) \;=\; \big(\mathcal{M}_G \circ E^{-1}\big)(x) \;=\; \frac{\tan \pi x}{\cos \pi\vartheta \tan \pi x \,+\, \sin \pi\vartheta} \tag{3.7}$$

$$g^{-1}(\xi) \;=\; \big(E \circ \mathcal{M}_{G^{-1}}\big)(\xi) \;=\; \frac{1}{\pi} \arctan\left[\frac{(\sin \pi\vartheta)\,\xi}{(\cos \pi\vartheta)\,\xi \,-\, 1}\right], \tag{3.8}$$

where $E : \partial D := \{z \in \mathbb{C} : |z| = 1\} \to \mathbb{T}$ is the bijection $\mathrm{e}^{\mathrm{i}\phi} \mapsto \frac{\phi}{2\pi}$, and the columns of G consists of eigenvectors of $\mathsf{E}(A_l)$.

Proof. By diagonalizing, we get $\mathsf{E}(A_l) = G\Lambda G^{-1}$ where

$$\Lambda \;=\; \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi\vartheta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\pi\vartheta} \end{bmatrix}, \quad G \;=\; \begin{bmatrix} 1 & -1 \\ \mathrm{e}^{-\mathrm{i}\pi\vartheta} & -\mathrm{e}^{\mathrm{i}\pi\vartheta} \end{bmatrix}, \quad G^{-1} \;=\; \frac{1}{2\mathrm{i}\sin \pi\vartheta} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi\vartheta} & -1 \\ \mathrm{e}^{-\mathrm{i}\pi\vartheta} & -1 \end{bmatrix}, \tag{3.9}$$

and ϑ is given in (3.6). From (3.9) we see that $\mathcal{M}_{G^{-1}}(\bar{\mathbb{R}}) = \partial D$. Since the matrix G is invertible, the property (3.1) implies that the associated Möbius transformation is also invertible. In particular, the restrictions $\mathcal{M}_G|_{\partial D}$ and $\mathcal{M}_G^{-1}|_{\bar{\mathbb{R}}} = \mathcal{M}_{G^{-1}}|_{\bar{\mathbb{R}}}$ are bijections mapping ∂D into $\bar{\mathbb{R}}$ and $\bar{\mathbb{R}}$ into ∂D, respectively. Using these observations we identify the coordinate transformation $g : \mathbb{T} \to \bar{\mathbb{R}}$ and its inverse $g^{-1} : \bar{\mathbb{R}} \to \mathbb{T}$ by regrouping as follows:

$$\begin{aligned} \mathcal{M}_{\mathsf{E}(A_l)} \;&=\; \mathcal{M}_G \circ \mathcal{M}_\Lambda \circ \mathcal{M}_{G^{-1}} \\ &=\; \big(\mathcal{M}_G \circ E^{-1}\big) \circ \big(E \circ \mathcal{M}_\Lambda \circ E^{-1}\big) \circ \big(E \circ \mathcal{M}_G^{-1}\big) \\ &=\; g \circ \lambda \circ g^{-1}\,, \end{aligned} \tag{3.10}$$

where λ equals the shift function on the right of (3.5).

In order to derive (3.7) and (3.8) the easiest way is to first solve g^{-1} using $E(z/z^*) = 2E(z) = \pi^{-1}\arctan\big[\Im(z)/\Re(z)\big]$:

$$x \;=\; g^{-1}(\xi) \;\equiv\; E\!\left(\frac{\mathrm{e}^{\mathrm{i}\pi\vartheta}\xi - 1}{\mathrm{e}^{-\mathrm{i}\pi\vartheta}\xi - 1}\right) \;=\; \pi^{-1}\arctan\left[\frac{\xi \sin(\pi\vartheta)}{\xi\cos(\pi\vartheta) - 1}\right].$$

The formula for g follows now by simply inverting the above function. □

Suppose $\xi \in \bar{\mathbb{R}}$ and $\xi' = \mathcal{M}_{\mathsf{E}(A_l)}(\xi)$. The important property of the new coordinates x is that even though the step $|\xi' - \xi|$ can be arbitrary large[3] regardless of how small w is, in the new coordinates every step $g^{-1}(\xi') - g^{-1}(\xi) = \vartheta$ is of size w. The next lemma says that this property remains true even when $\mathcal{M}_{\mathsf{E}(A_l)}$ is replaced by the random evolution $\mathcal{M}_{A_l}$.

Lemma 3.2. *Let $w > 0$ be fixed and let $g : \mathbb{T} \to \bar{\mathbb{R}}$ be the w-dependent coordinate tranformation (3.7). Then for any $b \in \,]0, \infty[$ the function*

$$f_b \;:=\; g \circ \mathcal{M}_A \circ g^{-1} : \mathbb{T} \to \mathbb{T} \qquad \textit{where} \qquad A \;\equiv\; A(b) \;:=\; \begin{bmatrix} 2 - \pi^2 w^2(1+b) & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.11}$$

is a bijection, that can be written as

$$f_b(x) \;=\; x \,+\, \vartheta \,+\, \Phi(x, b) \tag{3.12a}$$

$$f_b^{-1}(y) \;=\; y \,-\, \vartheta \,+\, \Phi(y - \vartheta, -b)\,, \tag{3.12b}$$

[3] Jumps $|\xi' - \xi|$ become arbitrary large as ξ approaches 0.

where the constant $\vartheta = w + \mathcal{O}(w^3)$ *is given in* (3.6) *and the smooth function* $\Phi : \mathbb{T}\times\,]0,\infty[\, \to \mathbb{T}$ *is specified by*

$$
\begin{aligned}
\Phi(x,b) \;&=\; \frac{1}{\pi}\arctan\left\{\frac{(\pi w/2)\big[1-\cos(2\pi x)\big]\,b}{\sqrt{1-(\pi w/2)^2}\;-\;(\pi w/2)\sin(2\pi x)\,b}\right\} && (3.13a)\\
&=\; \sin^2(\pi x)\Big[wb \;+\; w^2b^2(\pi/2)\sin(2\pi x) \;+\; w^3 b\, R_3(w,x,b)\Big]\,. && (3.13b)
\end{aligned}
$$

The remainder term $R_3 : [0,w_0]\times\mathbb{T}\times[b_-,b_+]$ *is a smooth and bounded function.*

The lemma says that in x-coordinates the system $\xi_n = \mathcal{M}_{A_n}(\xi_{n-1})$, $n\in\mathbb{N}$, and $\xi_0 = g^{-1}(x)$ is described by the following process on a circle. The proof which is just a mechanical calculation can be found in appendix A.1.

DEFINITION 3.3. *Let* $x\in\mathbb{T}$. *Markov process* $X^x \equiv (X^x_n : n\in\mathbb{N}_0)$ *on* $\mathbb{T}$ *is defined by setting*

$$
\begin{aligned}
X^x_n \;&=\; f_{B_n}(X^x_{n-1}) \qquad (n\in\mathbb{N})\\
X^x_0 \;&=\; x\,.
\end{aligned}
\tag{3.14}
$$

When the starting point x *is known from the context or its specific value is not relevant we write simply* X *and* X_n *instead of* X^x *and* X^x_n, *respectively.*

The main properties of $f_b(x)$ are best seen by expanding it into the power series w.r.t. w. Indeed, by using (3.6), (3.12a) and (3.13) one gets:

$$
\begin{aligned}
f_b(x) \;&=\; x + w + w\phi(x)\,b + w^2\psi(x)b^2 + \mathcal{O}(w^3)\,, && (3.15a)\\
\phi(x) \;&=\; \sin^2\pi x\,, && (3.15b)\\
\psi(x) \;&=\; \pi\sin^3\pi x\,\cos\pi x\,. && (3.15c)
\end{aligned}
$$

Let us denote $\Delta Z_k := Z_k - Z_{k-1}$ for a stochastic process (Z_k). By using the expansion (3.15) together with $\mathsf{E}(B_k) = 0$ and $B_k \ge b_- > -1$ the following qualitative properties of X emerge.

COROLLARY 3.4. *The process* X *has the following three useful properties:*

(i) *Uniform monotonicity:* $0 \;<\; (1+b_-)w + \mathcal{O}(w^2) \;\le\; \Delta X_k \;\le\; (1+b_+)w + \mathcal{O}(w^2)\,;$

(ii) $\mathcal{O}(w^1)$*-martingale property modulo constant shift:* $\mathsf{E}\big[\Delta X_k - w\big|\mathcal{F}_{k-1}\big] \;=\; X_{k-1} + \mathcal{O}(w^2)\,;$

(iii) *Uniform diffusion outside any neighborhood of zero: There are constants* $\alpha(\varepsilon),\beta>0$ *such that* $\mathsf{E}\big[(\Delta X_k - w)^2\big|X_{k-1}=x\big] \;\in\; [\alpha(\varepsilon)w^2, \beta w^2]$ *for* $|x|_{\mathbb{T}}\ge\varepsilon$.

Having found good coordinates $x = g(\xi)$ where $\xi_n = D_n/D_{n-1}$ evolves in w-sized steps in a relatively simple manner, our next step is to express the matrix elements of Q_n in terms of these new coordinates.

PROPOSITION 3.5. *Let* $v = [v_0\; v_{-1}]^{\mathrm{T}} \in \bar{\mathbb{R}}^2$ *with* $v_0 \ne 0$. *Then there is a constant* $w_0 > 0$ *such that for* $w\in\,]0,w_0]$ *the solution of* (2.3) *is*

$$
D_n(v) \;=\; v_0\cdot\Gamma^x_n\cdot\frac{\sin\pi X^x_n}{\sin\pi[\vartheta+\Phi(x,B_1)]} \qquad \textit{with} \quad x = g^{-1}(v_1/v_2)\,,
\tag{3.16}
$$

almost surely. Here the random amplitude $\Gamma^x_n : \Omega\to\,]0,\infty[$ *has an exponential representation*

$$
\Gamma^x_n \;=\; \exp\left[w\sum_{l=1}^{n} s(X^x_{l-1})B_l \;+\; w^2\sum_{l=1}^{n} r(X^x_{l-1})B_l^2 \;+\; \mathcal{O}(w^3 n)\right],
\tag{3.17}
$$

where the smooth functions $r, s : \mathbb{T} \to \mathbb{R}$ are specified by

$$s(x) \;=\; -\frac{\pi}{2}\sin 2\pi x\,, \tag{3.18a}$$

$$r(x) \;=\; \frac{\pi^2}{4}(\cos^2 2\pi x \;-\; \cos 2\pi x)\,. \tag{3.18b}$$

PROOF. Denote $D_n := D_n(v)$, $\xi_n = D_n/D_{n-1}$ and set $x := g^{-1}(\xi_0) \equiv g^{-1}(v_0/v_{-1})$. By definition (3.3) the process (ξ_n) is described in x-coordinates by the process (X_n^x). Set $X_n := X_n^x$ and use (3.7) to write

$$\xi_l \;=\; g \circ X_l \;=\; \mathcal{M}_G \circ E^{-1}(X_l) \;=\; \mathcal{M}_G(\mathrm{e}^{\mathrm{i}2\pi X_l})\,. \tag{3.19}$$

By using (3.9) to write out the Möbius transformation we obtain:

$$\mathcal{M}_G(\mathrm{e}^{\mathrm{i}\phi}) \;=\; \frac{\mathrm{e}^{\mathrm{i}\phi} - 1}{\mathrm{e}^{\mathrm{i}(\phi-\pi\vartheta)} - \mathrm{e}^{\mathrm{i}\pi\vartheta}} \;=\; \frac{\sin\frac{\phi}{2}}{\sin\bigl(\frac{\phi}{2} - \pi\vartheta\bigr)}\,.$$

By combining this with (3.19), reorganizing the resulting product and then using (3.12a) to write f in terms of Φ yields

$$\begin{aligned}
\frac{D_n}{v_0} \;&=\; \frac{\xi_n\xi_{n-1}\cdots\xi_1 v_0}{v_0} \;=\; \prod_{l=1}^{n}\frac{\sin\pi X_l}{\sin\pi(X_l-\vartheta)} \;=\; \frac{\sin\pi X_n}{\sin\pi(X_1-\vartheta)}\prod_{l=1}^{n-1}\frac{\sin\pi X_l}{\sin\pi(X_{l+1}-\vartheta)} \\
&=\; \frac{\sin\pi X_n}{\sin\pi[x+\Phi(x,B_1)]}\prod_{l=1}^{n-1}\frac{\sin\pi X_l}{\sin\pi[X_l+\Phi(X_l,B_{l+1})]}\,.
\end{aligned} \tag{3.20}$$

Here the possible extreme values $\xi_k \in \{0,\infty\}$ do not cause problems because we assumed $\xi_0 = v_0/v_{-1} \neq 0$ and (3.3) implies

$$\mathsf{P}\bigl(\xi_k \in \{0,\infty\} \text{ for some } k \in \mathbb{N}\big|\xi_0 \neq 0\bigr) \;=\; 0\,.$$

We must now show that the product of sin ratios in (3.20) equals the exponent Γ_n^x. Since, the terms in the product are all similar let us consider only one such factor. From (3.13b) one sees that $\Phi(x,b) = \mathcal{O}(w)$. This suggests expressing the denominators on the last line of (3.20) as power series of $\pi\Phi(x,b)$ around zero:

$$\begin{aligned}
\sin\pi(x+\Phi(x,b)) \;&=\; \sin\pi x\,\cos\pi\Phi(x,b) \;+\; \cos\pi x\,\sin\pi\Phi(x,b) \\
&=\; \sin\pi x\,\bigl\{1 - \frac{1}{2}\pi^2\Phi^2(x,b)\bigr\} \;+\; \pi\Phi(x,b)\cos\pi x \;+\; \mathcal{O}\bigl(\Phi^3(x,b)\bigr)\,.
\end{aligned} \tag{3.21}$$

The expression (3.13b) also shows that $\Phi^k(x,b)/\sin\pi x = \mathcal{O}(w^k)$ for $k \geq 1/2$. Thus using (3.21) to rewrite the denominators in (3.20) and then dividing the numerator and the denominator by $\sin\pi x$ yields the expression for geometric sum of variable $q = -\pi\Phi(x,b)\cot\pi x + \frac{\pi^2}{2}\Phi^2(x,b) + \mathcal{O}(w^3) = \mathcal{O}(w)$. Expanding this geometric sum gives the first line of

$$\begin{aligned}
\frac{\sin\pi x}{\sin\pi(x+\Phi(x,b))} \;&=\; 1 \;-\; \pi\Phi(x,b)\cot\pi x + \frac{\pi^2}{2}\Phi^2(x,b) \;+\; \pi^2\Phi^2(x,b)\cot^2\pi x \;+\; \mathcal{O}(w^3) \\
&=\; 1 \;-\; w\frac{\pi}{2}\sin 2\pi x\; b \;+\; w^2\frac{\pi^2}{8}(1-\cos 2\pi x)^2\, b^2 + \mathcal{O}(w^3)\,,
\end{aligned}$$

while the last line follows from (3.13b) and trigonometric double angle formulae. By using $1 + z = \exp\circ\ln(1+z) = \exp\bigl[z - \frac{1}{2}z^2 + \mathcal{O}(z^3)\bigr]$, with $|z| \leq Cw_0$, for the last expression we get

$$\frac{\sin\pi x}{\sin\pi(x+\Phi(x,b))} \;=\; \exp\Bigl[-w(\pi/2)\sin 2\pi x\; b \;+\; w^2(\pi/2)^2\bigl(\cos^2 2\pi x - \cos 2\pi x\bigr)b^2 + \mathcal{O}(w^3)\Bigr]\,.$$

Identifying functions s and r on the right side and then applying this bound term by term for the product in (3.20) yields the expression on the right side of (3.17). $\square$

It is worth remarking that the proposition does not apply directly for $v \in \mathbb{C}^2$ since it relies on Lemmas 3.1 and 3.2 which apply only when (ξ_n) takes values on $\mathbb{R}$. Of course, by the linearity of the system (2.3) one still has $D_n(v_R + \mathrm{i}v_I) = D_n(v_R) + \mathrm{i}D_n(v_I)$ for any $v_R, v_I \in \mathbb{R}^2$. The next corollary shows that the generic choice $D_n(v)$ with $v = e_k$, $k = 1, 2$, is often a convenient choice as $D(e_2)$ can be treated as a perturbation of $D(e_1)$.

COROLLARY 3.6. *There is a constant $w_0 > 0$ such that for $w \in \,]0, w_0]$:*

$$D_n(e_1) \;=\; \Gamma_n^{\vartheta} \cdot \frac{\sin \pi X_n^{\vartheta}}{\sin \pi[\vartheta + \Phi(\vartheta, B_1)]} \;\sim\; w^{-1}\Gamma_n^{\vartheta} \cdot \sin \pi X_n^{\vartheta}\,, \tag{3.22a}$$

$$D_n(e_2) \;=\; \Gamma_n^{0} \cdot \frac{\sin \pi X_n^{0}}{\sin \pi[\vartheta + \Phi(\vartheta, B_2)]} \;\sim\; w^{-1}\Gamma_n^{0} \cdot \sin \pi X_n^{0}\,. \tag{3.22b}$$

PROOF. By (3.8) we get $g^{-1}(\xi_0) = g^{-1}(1/0) = \vartheta$ and thus (3.22a) follows directly from Proposition 3.5. In order to prove (3.22b) one can not directly apply the proposition since the first component of e_2 is zero. However, from (2.3) one sees that $[D_1(e_2)\;\, D_0(e_2)]^{\mathrm{T}} = [-1\;\, 0]^{\mathrm{T}} = -e_1$ and $D_n(-v) = -D_n(v)$. Thus, by defining $\theta : \Omega \to \Omega$ by $\theta\omega = (b_2, b_3, \dots)$ for $\omega = (b_1, b_2, \dots)$ and denoting the associated pullback θ_* on random variables Z by $\theta_* Z(\omega) = Z(\theta\omega)$, one can write

$$D_n(e_2) \;=\; -\theta_* D_{n-1} \;=\; \theta_*\Gamma_{n-1}^{\vartheta} \cdot \frac{\sin \pi\, \theta_* X_{n-1}^{\vartheta}}{\sin \pi[\vartheta + \Phi(\vartheta, \theta_* B_1)]}\,, \tag{3.23}$$

where by the definition:

$$\theta_*\Gamma_{n-1}^{\vartheta} \;=\; \exp\left[w \sum_{l=1}^{n-1} s(\theta_* X_{l-1}^{\vartheta})\, \theta_* B_l \;+\; w^2 \sum_{l=1}^{n-1} r(\theta_* X_{l-1}^{\vartheta})(\theta_* B_l)^2 \;+\; \mathcal{O}(w^3 n) \right]. \tag{3.24}$$

Now, since $\Phi(0, b) = 0$ it follows that $X_1^0 = f_{B_1}(0) = \vartheta + \Phi(0, B_1) = \vartheta = \theta_* X_0^{\vartheta}$ regardless of the value of B_1. But $(X_n^0 : n \in \mathbb{N})$ and $(\theta_* X_{n-1}^{\vartheta} : n \in \mathbb{N})$ also satisfy the same recursion relations for $n \geq 2$ and therefore $\theta_* X_n^{\vartheta} = X_{n+1}^0$, $n \in \mathbb{N}_0$. Also, by definition $\theta_* B_l(\omega) = b_{l+1} = B_{l+1}(\omega)$. Thus we may replace $\theta_* X_{l-1}^{\vartheta}$ with X_l^0 and write $\theta_* B_l = B_{l+1}$ in (3.23) and (3.24). Moreover, if we also reindex the sums in (3.24) we obtain an exponential representation for $\theta_*\Gamma_{n-1}$ that is up to a missing first terms $ws(X_0^0)B_1$ and $w^2 r(X_0^0)B_1^2$ equal to Γ_n^0. However, these missing terms are both zero due to the "coincidence" $s(0) = r(0) = 0$, and thus we get $\theta_*\Gamma_{n-1} = \Gamma_n^0$. This proves (3.22b). □

3.2 Joint behavior

In order to prove $n^{-3/2} \lesssim J_n$ we analyze the current density j_n defined in (2.7). This leads us to consider the properties of the quadruple $(X_n^{\vartheta}, X_n^0, \Gamma_n^{\vartheta}, \Gamma_n^0)$. Since $X_0^{\vartheta} - X_0^0 = \vartheta \sim w$ one can consider X_n^0 and Γ_n^0 as perturbations around X_n^{ϑ} and Γ_n^{ϑ}, respectively. Based on this simple idea one proves the following.

LEMMA 3.7. *Let us treat X^x, $x \in \mathbb{R}$ as real valued processes. Then for all $n \in \mathbb{N}$ and $w \in \,]0, w_0]$:*

$$X_n^{\vartheta} - X_n^0 \;= w\, \mathrm{e}^{M_n + L_n + \mathcal{O}(w^2 n)} \tag{3.25}$$

$$\Gamma_n^0 / \Gamma_n^{\vartheta} \;= \mathrm{e}^{K_n + \mathcal{O}(w + w^2 n)}\,, \tag{3.26}$$

where $(M_n), (L_n), (K_n)$ are $\mathbb{R}$-valued $\mathbb{F}$-martingales such that $M_0 = L_0 = K_0 = 0$ and $n \in \mathbb{N}$:

$$\Delta M_n \;=\; w\phi'(X_{n-1}^{\vartheta})B_n \tag{3.27}$$

$$\Delta L_n \;=\; w^2 \mathrm{e}^{M_{n-1} + L_{n-1} + \mathcal{O}(w^2 n)} H_{n-1} B_n \tag{3.28}$$

$$\Delta K_n \;=\; w^2 \mathrm{e}^{M_{n-1} + L_{n-1} + \mathcal{O}(w^2 n)} U_{n-1} B_n\,. \tag{3.29}$$

The processes (H_n) and (U_n) are $\mathbb{F}$-adapted and bounded such that:

$$\sup\left\{|H_n|,\, |U_n|,\, w^{-1}|\Delta L_n|,\, w^{-1}|\Delta K_n| \,:\, n \in \mathbb{N}\right\} \,\leq\, C\,. \tag{3.30}$$

Proof. From (3.13b) and (3.15b) one sees that $\Phi(x,b) = w\phi(x)b + w^2 R_2(x,b)$ where R_2 is a smooth and bounded function. Using (3.12a) we get

$$\begin{aligned} f_b(x) \,-\, f_b(x-z) \;&=\; z \,+\, \Phi(x,b) - \Phi(x-z,b) \\ &=\; \left\{1 \,+\, w\frac{\phi(x)-\phi(x-z)}{z}\,b \,+\, w^2\frac{R_2(x,b)-R_2(x-z,b)}{z}\right\} z\,, \end{aligned} \tag{3.31}$$

for any $z \in \mathbb{R}$. By the mean value theorem there are function $\zeta_1(x,z), \zeta_2(x,z,b) \in [x-z,x]$ such that for any $x \in \mathbb{R}$, $z \geq 0$ and $b \in [b_-, b_+]$ we have

$$\begin{aligned} f_b(x) \,-\, f_b(x-z) \;&=\; \left\{1 \,+\, w\phi'(x)b \,-\, wz\,\frac{1}{2}\phi''(\zeta_1(x,z))b \,+\, w^2\partial_x R_2(\zeta_2(x,z,b),b)\right\} z \\ &=\; \exp\!\left[w\phi'(x)b \,-\, wz\,\frac{1}{2}\phi''\circ\zeta_1(x,z)b \,+\, \mathcal{O}(w^2)\right] z\,. \end{aligned} \tag{3.32}$$

Now, set

$$\Theta_n \;:=\; (X_n^\vartheta - X_n^0)/w \qquad \text{and} \qquad H_n \;:=\; -\frac{1}{2}\phi''\circ\zeta_1(X_n^\vartheta, w\Theta_n)\,, \tag{3.33}$$

Then (3.32) and (3.14) yield

$$\begin{aligned} \Theta_n \;&=\; \frac{1}{w}\{f_{B_n}(X_{n-1}^\vartheta) \,-\, f_{B_n}(X_{n-1}^\vartheta - w\Theta_{n-1})\} \\ &=\; \exp\!\left[w\phi'(X_{n-1}^\vartheta)B_n \,-\, w^2\Theta_{n-1}\frac{1}{2}\phi''\circ\zeta_1(X_{n-1}^\vartheta, w\Theta_{n-1})B_n \,+\, \mathcal{O}(w^2)\right]\cdot\Theta_{n-1} \\ &=\; \exp\!\left[w\sum_{j=1}^n \phi'(X_{j-1}^\vartheta)B_j \,+\, w^2\sum_{j=1}^n \Theta_{j-1}\,H_{j-1}B_j \,+\, \mathcal{O}(w^2 n)\right]\cdot\Theta_0\,. \end{aligned} \tag{3.34}$$

By using (3.27) and (3.28) we identify the two sums inside the exponent in (3.34) as M_n and L_n, respectively. Together with $\Theta_0 = (X_0^\vartheta - X_0^0)/w = \vartheta/w = 1 + \mathcal{O}(w^2)$ this gives $\Theta_n = \mathrm{e}^{M_n+L_n+\mathcal{O}(w^2 n)}$ and by the definition (3.33) this equals (3.25). Moreover, $w^{-1}\Delta L_{n+1} = w\Theta_n H_n B_{n+1}$, where using (3.32), (3.33) and the definition of ζ_1 we get

$$w\Theta_n H_n \;=\; \frac{\phi(X_n^\vartheta) - \phi(X_n^0)}{X_n^\vartheta - X_n^0} \,-\, \phi'(X_n^\vartheta) \;=:\; \phi'(\zeta_0) \,-\, \phi'(X_n^\vartheta)\,,$$

for some $\zeta_0 \in [X_n^0, X_n^\vartheta]$, and therefore $w^{-1}|\Delta L_{n+1}| \leq 2\|\phi'\|_\infty \cdot \max\{-b_-, b_+\} =: C$.

In order to prove (3.26) we use again the mean value theorem to write

$$s(X_n^0) \;=\; s(X_n^\vartheta - w\Theta_n) \;=\; s(X_n^\vartheta) \,-\, w\Theta_n\cdot s'\circ\zeta_3(X_n^\vartheta, w\Theta_n)\,, \tag{3.35}$$

where $X_n^\vartheta - w\Theta_n \leq \zeta_3(X_n^\vartheta, w\Theta_n) \leq X_n^\vartheta$. Using this in (3.17) yields

$$\begin{aligned} \Gamma_n^0 \;&=\; \exp\left[w\sum_{l=1}^n s(X_{l-1}^0)B_l \,+\, \mathcal{O}(w^2 n)\right] \\ &=\; \exp\left[w\sum_{l=1}^n s(X_{l-1}^\vartheta)B_l \,-\, w^2\sum_{l=1}^n \Theta_{l-1}\cdot s'\circ\zeta_3(X_{l-1}^\vartheta, w\Theta_{l-1})B_l \,+\, \mathcal{O}(w^2 n)\right] \\ &=:\; \Gamma_n^\vartheta\,\mathrm{e}^{K_n\,+\,\mathcal{O}(w+w^2 n)}\,. \end{aligned}$$

Above, we have identified $U_n = -s'\circ\zeta_3(X_n^\vartheta, w\Theta_n)$ in (3.29). Finally, by equation (3.35) $w^{-1}\Delta K_{n+1} = w\Theta_n U_n B_{n+1} = [s(X_n^0) - s(X_n^\vartheta)]\,B_{n+1}$. Since s is a bounded function (3.18a) this implies $w^{-1}|\Delta K_n| \leq C$. ☐

4 Expectation of $1/\Gamma_n$

In this section we prove the following result.

PROPOSITION 4.1. *For sufficiently small $w_0 \sim 1$ there exists $\alpha \equiv \alpha(w_0) > 0$ such that for $n \in \mathbb{N}$,*

$$\sup_{x \in \mathbb{T}} \mathsf{E}\big(1/\Gamma_n^x\big) \;\lesssim\; \mathrm{e}^{-\alpha w^2 n}\,, \qquad w \in \,]0, w_0]\,. \tag{4.1}$$

The content of this result is best understood by using (3.17) to write $1/\Gamma_n$ as exponent $\mathrm{e}^{-R_n w^2 n \,+\, w n^{1/2} S_n \,+\, \mathcal{O}(w^3 n)}$, where the normalized random variables

$$S_n \;=\; \frac{-1}{n^{1/2}} \sum_{k=1}^{n} s(X_{k-1}) B_k \qquad \text{and} \qquad R_n \;=\; \frac{1}{n} \sum_{k=1}^{n} r(X_{k-1}) B_k^2\,,$$

are in average of order 1. Our proof of Proposition 4.1 consists of two steps which both rely on the fact that during any consecutive sequence of $\lfloor 1/w \rfloor$ steps the random set $\{X_j(w) : j = k, \dots, k + \lfloor 1/w \rfloor\}$, $k \in \mathbb{N}$, typically samples $\mathbb{T}$ evenly. First, Lemma 4.4 is used to shows that $R_n \equiv R_n(w)$ can be replaced by the constant $\gamma(w)/w^2$ without introducing too large errors in $\mathsf{E}(1/\Gamma_n)$ provided $wn \to \infty$. Here

$$\gamma(w) \;=\; \left\{ \mathsf{E}(B_1^2) \cdot \int_{\mathbb{T}} r(x) \mathrm{d}x \right\} w^2 \,+\, \mathcal{O}(w^3) \;=\; \frac{\pi^2 \mathsf{E}(B_1^2)}{8} w^2 + \mathcal{O}(w^3)\,, \tag{4.2}$$

is the Lyapunov exponent associated to the norm of Q_n in (2.4). Secondly, the uniform monotonicity (property (i) of Corollary 3.4) of the process X is used to bound the conditional variance (see (4.3)) of the martingale $n^{1/2} S_n$ so that Freedman's powerful exponential martingale bound, i.e., Lemma 4.2, can be applied to obtain a bound $\mathsf{E}\mathrm{e}^{w n^{1/2} S_n} \le \mathrm{e}^{\beta w^2 n}$, where $\gamma(w)/w^2 - \beta =: \alpha \sim 1$.

The following lemma provides two powerful exponential martingale bounds due to Freedman [11] and Azuma [3].

LEMMA 4.2. *Let (M_i) be a $(\mathcal{F}_i)$-martingale, and define a process (V_n) by setting $V_0 = 0$ and*

$$V_n \;:=\; \sum_{i=1}^{n} \mathsf{E}\big[(M_i - M_{i-1})^2 \big| \mathcal{F}_{i-1}\big]\,, \qquad n \in \mathbb{N}\,. \tag{4.3}$$

Suppose there exists a constant m and a sequence $(v_n) \subset [0, \infty[$ such that $|M_n - M_{n-1}| \le m$ and $V_n \le v_n$ for all $n \in \mathbb{N}$. Then for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\mathsf{E}\,\mathrm{e}^{t M_n} \;\le\; \begin{cases} \mathrm{e}^{\kappa_m(t) v_n}\,, & \text{"Freedman's bound";} \\ \mathrm{e}^{\frac{t^2}{2} m^2 n}\,, & \text{"Azuma's bound";} \end{cases} \tag{4.4}$$

where

$$\kappa_m(t) \;=\; \frac{\mathrm{e}^{mt} - 1 - mt}{m^2} \;\le\; \frac{t^2}{2} \,+\, \frac{m}{6} \mathrm{e}^{m|t|} |t|^3\,. \tag{4.5}$$

For the convenience of readers the proofs of these bounds are included in Appendix A.2. The next inequality (4.6) is often referred as Azuma's inequality.

COROLLARY 4.3. *Suppose (M_k) satisfies the hypothesis of Lemma 4.2. Then for any $n \in \mathbb{N}$ and $r > 0$:*

$$\mathsf{P}(|M_n| \ge r) \;\le\; 2\,\mathrm{e}^{-\frac{r^2}{2m^2 n}}\,. \tag{4.6}$$

Proof. The proof follows by using Markov's inequality: $\mathsf{P}(|M_n| \geq r) = \mathsf{P}(M_n \geq r) + \mathsf{P}(-M_n \geq r) \leq \mathrm{e}^{-sr}\mathsf{E}\,\mathrm{e}^{sM_n} + \mathrm{e}^{-sr}\mathsf{E}\,\mathrm{e}^{-sM_n}$, and then use Azuma's bound (4.4) with $t = r/(m^2 n)$. $\square$

Lemma 4.4. *Suppose u is a Lipshitz-function on $\mathbb{T}$, i.e., there is a constant $L_u > 0$ such that for all $x, y \in \mathbb{T}$: $|u(x) - u(y)| \leq L_u |x-y|_{\mathbb{T}}$. Then:*

$$\sup_{x\in\mathbb{T}} \mathsf{E}\left\{ \left| w \sum_{j=0}^{\lfloor 1/w \rfloor} u(X_j^x) \;-\; \int_{\mathbb{T}} u(y)\mathrm{d}y \right|^p \right\} \;\leq\; C_p L_u^p w^{p/2}\,, \tag{4.7}$$

where C_p does not depend on u.

Proof. Fix x and set $X := X^x$ and $I_j := [x + w\,(j-1), x + w\,j[$. Define for each j some $\tilde{x}_j \in I_j$ by requiring $\int_{I_j} u(x)\mathrm{d}x = w\,u(\tilde{x}_j)$, and set $\bar{x}_j := \mathsf{E}(X_j)$. The properties (3.15) of the chain X imply $|\bar{x}_j - \tilde{x}_j| \leq w$ for all $j \leq \lfloor 1/w \rfloor$. By writing the integral on the left side of (4.7) as a sum over $u(\tilde{x}_j)$ and then applying the Lipshitz-property of u one gets

$$\mathsf{E}\left\{ \left| w \sum_{j=0}^{\lfloor 1/w \rfloor} \big[u(X_j) - u(\tilde{x}_j)\big] \right|^p \right\} \;\leq\; L_u^p\, w^p \sum_{j_1,\dots,j_p} \mathsf{E}\left\{ \prod_{l=1}^{p} |X_{j_l} - \tilde{x}_{j_l}| \right\}. \tag{4.8}$$

Now, $X_j = x + wj + w^{1/2}M_j + \mathcal{O}(w)$ with $M_j = w^{1/2}\sum_{i=1}^{j}\phi(X_{i-1})B_i$ uniformly for any $0 \leq j \leq \lfloor 1/w \rfloor$. This means $X_j - \tilde{x}_j = w^{1/2}(M_j + \mathcal{O}(w^{1/2}))$. By applying the generalized Hölder's inequality one has,

$$\begin{aligned} \mathsf{E}\left\{ \prod_{l=1}^{p} |X_{j_l} - \tilde{x}_{j_l}| \right\} \;&=\; w^{p/2}\mathsf{E}\left\{ \prod_{l=1}^{p} \big|M_{j_l} + \mathcal{O}(w^{1/2})\big| \right\} \\ &\leq\; w^{p/2} \left(\prod_{l=1}^{p} \mathsf{E}\Big\{ \big|M_{j_l} + \mathcal{O}(w^{1/2})\big|^p \Big\} \right)^{1/p}. \end{aligned} \tag{4.9}$$

The last expectations of (4.9) can be bounded with Azuma's inequality (4.6). Indeed, $|M_j - M_{j-1}| \leq w^{1/2}\max(-b_-, b_+)\|\phi\|_\infty \equiv C w^{1/2}$ for each j. This implies $\mathsf{P}\big(|M_j| \in [k, k+1[\big) \leq 2\mathsf{P}(|M_j| \geq k) \leq 2\mathrm{e}^{-k^2/(2C^2 w\lfloor 1/w \rfloor)} = \mathrm{e}^{-k^2/C'}$ which, in turn, yields

$$\mathsf{E}\Big\{ \big|M_j + \mathcal{O}(w^{1/2})\big|^p \Big\} \;\leq\; \sum_{k=0}^{\infty} (k + 1 + \mathcal{O}(w^{1/2}))^p \mathsf{P}\big(|M_j| \in [k, k+1[\big) \;\leq\; 2\sum_{k=0}^{\infty} k^p \mathrm{e}^{-k^2/C'} \;=: C_p\,,$$

Since this bound holds uniformly for all $j = 0, 1, \dots, \lfloor 1/w \rfloor$ we may apply it term by term in (4.9). Using the resulting bound again term by term in (4.8) yields the bound (4.7). $\square$

Proof of Proposition 4.1. Since $\Gamma_n(w) \geq C$ for $wn \sim 1$ it is enough to show $\mathsf{E}(1/\Gamma_n^x) \leq C\,\mathrm{e}^{-\alpha w^2 n}$ for $n = \lfloor 1/w \rfloor m$, $m \in \mathbb{N}$. Since $\Delta X_n \geq Cw$ we may for the same reason fix some arbitrary starting point $x \in \mathbb{T}$ and denote X_n^x and Γ_n^x by X_n and Γ_n, respectively. We begin the proof by decomposing the second sum in the exponent of (3.17) into the double sum

$$w^2 \sum_{i=1}^{n} r(X_{i-1})B_i^2 \;=\; w \sum_{k=1}^{m} w \sum_{i=i_{k-1}+1}^{i_k} r(X_{i-1})B_i^2 \;=\; w \sum_{k=1}^{m} \gamma(X_{i_{k-1}}) \;+\; w \sum_{k=1}^{m} Z_k\,, \tag{4.10}$$

where $i_k = \lfloor 1/w \rfloor k + 1$, $k = 1, 2, \dots, m$ is roughly the time the averaged process $\bar{x}_j := \mathsf{E}_x(X_j) = x + wj + \mathcal{O}(w^3 j)$ has passed its starting point k^{th} time. In the rightmost expression of (4.10) we have further divided the inner sums into the conditional expectations and the fluctuation parts:

$$Z_k := w \sum_{i=i_{k-1}+1}^{i_k} r(X_{i-1})B_i^2 - \gamma(X_{i_{k-1}}) \tag{4.11a}$$

$$\gamma(y) := \mathsf{E}\Biggl\{ w \sum_{i=1}^{\lfloor 1/w \rfloor} r(X_{i-1}^y)B_i^2 \Biggr\}. \tag{4.11b}$$

The motivation behind the decomposition (4.10) is twofold. First, Lemma 4.4 tells us that the function γ is almost constant for small w, and especially

$$\gamma(y) = \mathsf{E}(B^2)\,\mathsf{E}\Biggl\{ w \sum_{i=1}^{\lfloor 1/w \rfloor} r(X_{i-1}^y) \Biggr\} \geq \mathsf{E}(B^2) \int_{\mathbb{T}} r(z)\mathrm{d}z - \beta_0 w^{1/2} =: \tilde{\gamma}_-, \tag{4.12}$$

where $\beta_0 > 0$ is a finite constant that does not depend on y. Here the first equality follows from $\mathsf{E}\bigl(r(X_{i-1})B_j\bigr) = \mathsf{E}(B^2)\,\mathsf{E}(r^2(X_i))$, while the last expression comes from Lemma 4.4 with $p = 1$ and $L_u := \|r'\|_\infty$. Using (4.12) to bound each term $\gamma(X_{i_{k-1}})$ in (4.10) yields the bound:

$$\mathsf{E}\bigl(1/\Gamma_n\bigr) \leq \mathrm{e}^{-\gamma_- w^2 n} \mathsf{E}\exp\Biggl[-w \sum_{i=1}^{n} s(X_{i-1})B_i - w \sum_{k=1}^{m} Z_k \Biggr], \quad \text{with} \quad \gamma := \tilde{\gamma} + \mathcal{O}(w), \tag{4.13}$$

where the $\mathcal{O}(w^3 n)$-term inside the exponent (3.17) of Γ_n has been also absorbed into the constant γ_-.

The second property of the decomposition (4.10) is that $(Z_k : k \in \mathbb{N})$ constitutes a sequence of bounded martingale increments in the *sparse filtration* $\mathbb{F}' = (\mathcal{F}'_k)$, $\mathcal{F}'_k := \mathcal{F}_{i_k} \equiv \sigma(B_1, B_2, \ldots, B_{i_k})$: the boundedness of Z_k is obvious as it is an average of $\lfloor 1/w \rfloor$ uniformly bounded increments, while the martingale property holds, since X is Markov:

$$\mathsf{E}\Biggl(w \sum_{i=i_{k-1}+1}^{i_k} r(X_{i-1})B_i^2 \Bigg| \mathcal{F}'_{k-1} \Biggr)(\omega) = \mathsf{E}\Biggl\{ w \sum_{i=1}^{\lfloor 1/w \rfloor} r\Bigl(X_{i-1}^{X_{i_{k-1}}(\omega)} \Bigr) B_i^2 \Biggr\} \equiv \gamma(X_{i_{k-1}}(\omega)),$$

for a.e. $\omega \in \Omega$. We want to consider both sums in the right side of (4.13) as martingales. Since this is not possible under the same expectation we apply Hölder's inequality to divide the expectation into the product of separate expectations

$$\mathsf{E}(1/\Gamma_n) \leq \mathrm{e}^{-\gamma_- w^2 n} \Biggl\{ \mathsf{E}\exp\Biggl[-pw \sum_{i=1}^{n} s(X_{i-1})B_i \Biggr] \Biggr\}^{1/p} \Biggl\{ \mathsf{E}\exp\Biggl[-p'w \sum_{k=1}^{m} Z_k \Biggr] \Biggr\}^{1/p'}, \tag{4.14}$$

where $p, p' \geq 1$ and $1/p + 1/p' = 1$. We can now bound both of these expectations with the help of Lemma 4.2. Azuma's exponential bound (4.4) is sufficient for the second factor: if $|Z_k| \leq C_Z$, then

$$\Biggl\{ \mathsf{E}\exp\Biggl[-p'w \sum_{k=1}^{m} Z_k \Biggr] \Biggr\}^{1/p'} \leq \Biggl\{ \exp\Biggl[\frac{(-p'w)^2}{2} C_Z^2 \lfloor nw \rfloor \Biggr] \Biggr\}^{1/p'} \leq \mathrm{e}^{\beta_2 p' w^3 n}, \tag{4.15}$$

for some constant β_2.

In order to handle the first expectation of (4.14) we note that the martingale (M_j), defined by $\Delta M_j := s(X_{j-1})B_j$, $j \in \mathbb{N}$ and $M_0 = 0$, has bounded increments. Moreover, since $\mathsf{E}[(\Delta M_i)^2 | \mathcal{F}_{i-1}] = \mathsf{E}(B^2) \cdot s^2(X_{i-1})$, we see that for sufficiently small $\varepsilon > 0$:

$$V_n := \sum_{i=1}^{n} \mathsf{E}\bigl[(M_i - M_{i-1})^2 \big| \mathcal{F}_{i-1} \bigr] = \mathsf{E}(B^2) \sum_{i=1}^{n} s^2(X_{i-1}) \leq (1-\varepsilon)\mathsf{E}(B^2)\|s\|_\infty^2 n.$$

In order to get the last bound above, one uses the property (i) of Corollary 3.4, the continuity of s and $s(0)=0$, to conclude that there must exist $\varepsilon>0$ such that

$$\big|\{0\le i\le n-1 : |s^2(X_i)|\le \|s\|_\infty^2/2\}\big| \;\ge\; 2\varepsilon n\,.$$

This, by definition, implies the bound of V_n above. Applying Freedman's bound of Lemma 4.2 with $v_n := \mathsf{E}(B^2)(1-\varepsilon)\|s\|_\infty^2 n$ and $|M_i - M_{i-1}|\le C_M =: m$ yields

$$\begin{aligned}\Big\{\mathsf{E}\exp\Big[(-wp)\sum_{i=1}^n s(X_{i-1})B_i\Big]\Big\}^{1/p} \;&\le\; \Big\{\exp\Big[\kappa_{C_M}(-wp)\mathsf{E}(B^2)(1-\varepsilon)\|s\|_\infty^2 n\Big]\Big\}^{1/p} \\ &\le\; \mathrm{e}^{\frac{1}{2}pw^2(1-\varepsilon)\mathsf{E}(B^2)\|s\|_\infty^2 n+\beta_1 p^2 w^3 n}\,,\end{aligned} \tag{4.16}$$

where $\beta_1 > (1/6)(1-\varepsilon)\mathsf{E}(B^2)C_M\mathrm{e}^{C_M pw}\sim 1$.

Plugging (4.16) and (4.15) along with the estimate (4.12) for γ_- into (4.14) results into the total bound

$$\mathsf{E}(1/\Gamma_n) \;\le\; \mathrm{e}^{-\mathsf{E}(B^2)\left\{\int_{\mathbb{T}} r(y)\mathrm{d}y\,-\,p(1-\varepsilon)\frac{\|s\|_\infty^2}{2}\right\}w^2 n+\beta_0 w^{5/2}n+\beta_1 p^2 w^3 n+\beta_2 p' w^3 n + Cw^3 n}\,. \tag{4.17}$$

Here the term inside curly brackets would disappear if $p=1, \varepsilon=0$ because $\int_{\mathbb{T}} r(y)\mathrm{d}y = \|s\|_\infty^2/2=\pi^2/8$. However, since $\varepsilon>0$ we can take $p>1$ such that it remains positive. However, by taking w_0 sufficiently small the last three terms, regardless of the size of p' or $\beta_1,\beta_2,\beta_3,C$, can be made arbitrary small compared to the first part. $\square$

5 Potential theory

This section is devoted to the statement and the proof of Proposition 5.1 below. The derivation of the inequalities (5.1a) and (5.1b) constitutes a relatively classical problem in potential theory for Markov chains. However, it does not seem possible to apply classical results (see e.g. [6] and [7]), since the chain X is neither reversible, nor uniformly diffusive. In particular, little appears to be known on lower bounds of the type (5.1b) for non-reversible Markov chains. Results for Markov chains on a lattice [16], or for differential equations in non-divergence form [10], do not adapt straightforwardly (and maybe not at all) to our case. Instead, since we consider only the case $w\to 0$, it has been possible to treat the left hand side of (5.1a) and (5.1b) as a perturbation of quantities that can be computed explicitly. We are then able to handle both of these bounds with a single method.

Proposition 5.1. *Let $\kappa>0$, and let $h\in \mathrm{C}^1(\mathbb{T})$. There exist $K, K', w_0>0$ such that, for every $w\in\,]0,w_0]$, for every function $u\in \mathrm{L}^1(\mathbb{T};\mathbb{R}_+)$, for every $x\in\mathbb{R}$, and for every $n\in\mathbb{N}$, one has*

$$\mathsf{E}\big(\mathrm{e}^{w\sum_{k=1}^n h(X_{k-1}^x)B_k}\,u(X_n^x)\big) \;\le\; \frac{K}{w\sqrt{n}}\int_{\mathbb{T}} u(y)\mathrm{d}y \qquad (wn\ge\kappa,\; w^2 n\le 1)\,, \tag{5.1a}$$

$$\mathsf{E}\big(\mathrm{e}^{w\sum_{k=1}^n h(X_{k-1}^x)B_k}\,u(X_n^x)\big) \;\ge\; K'\int_{\mathbb{T}} u(y)\mathrm{d}y \qquad (1/2\le w^2 n\le 1)\,. \tag{5.1b}$$

Before starting the proof let us make a few of definitions: First, for $A\subset\mathbb{T}$ and $1\le p\le\infty$ we define the space

$$\mathrm{L}^p_A(\mathbb{T}) \;:=\; \{u\in \mathrm{L}^p(\mathbb{T}) : \mathrm{supp}(u)\subset A\}\,.$$

Secondly, let S be a continuous operator from $\mathrm{L}^p(\mathbb{T})$ to $\mathrm{L}^q(\mathbb{T})$, for $1\le p,q\le\infty$, and denote the associated operator norm by $\|S\|_{p\to q}=\sup\{\|Su\|_q : u\in \mathrm{L}^p(\mathbb{T}),\ \|u\|_p\le 1\}$.

The content of Proposition 5.1 is twofold. First, it describes the approach to equilibrium of the chain X. To see this, let us consider the case $h=0$, and let us take some subset $A\subset\mathbb{T}$.

Equation (5.1a) implies that $\mathsf{P}(X_n^x \in A) \lesssim \max\{1/(w\sqrt{n}), 1\}\mathrm{Leb}(A)$ when $wn \geq \kappa$, whereas (5.1a) and (5.1b) imply that $\mathsf{P}(X_n^x \in A) \sim \mathrm{Leb}(A)$ when $w^2 n \geq 1/2$. This is obvious when $w^2 n \leq 1$. But, if $w^2 n > 1$, one can write $n = n_1 + n_2$ such that $\lceil n_2 w^2 \rceil = 1$, and

$$\mathsf{E}u(X_n^x) \;=\; \int_{\mathbb{T}} \mathsf{E}\big(u(X_n^x)|X_{n_1}^x = y\big)\, \mathsf{P}(X_{n_1}^x \in \mathrm{d}y)\,.$$

The result follows since, if $y \in \mathbb{T}$, one has $\mathsf{E}\big[u(X_n^x)\big|X_{n_1}^x = y\big] = \mathsf{E}u(X_{n_2}^y) \sim \|u\|_1$.

Secondly, Proposition 5.1 asserts that the result obtained for $h = 0$ is not destroyed when some specific perturbation is added ($h \neq 0$). If $h \neq 0$ but if $u = 1$, results (5.1a) and (5.1b) are trivial. Indeed, by Azuma's inequality (4.6), one finds some $C > 0$ such that, for every $n \in \mathbb{N}$ and for every $a > 0$, one has

$$\mathsf{P}\Big(\mathrm{e}^{-a} \leq \mathrm{e}^{w\sum_{k=1}^n h(X_{k-1}^x)B_k} \leq \mathrm{e}^{a}\Big) \;\geq\; 1 \,-\, 2\mathrm{e}^{-\frac{Ca^2}{w^2 n}}\,.$$

So, in general, one sees that the rare events where $\mathrm{e}^{w\sum_{k=1}^n h(X_{k-1}^x)B_k}$ is very large or very close to zero may essentially be neglected.

In the sequel, one assumes that

(A1) $\kappa > 0$ and $h \in \mathrm{C}^1(\mathbb{T})$ are given,

(A2) $w \in \,]0, w_0]$, where w_0 is small enough to make all our assertions valid.

All the constants introduced below may depend on κ and h.

In order to prove Proposition 5.1, let us introduce a continuous operator T on $\mathrm{L}^p(\mathbb{T})$, $1 \leq p \leq \infty$, by setting

$$Tu(x) \;:=\; \mathsf{E}\big[(1 + wh(x)B)\, u \circ f_B(x)\big] \;=\; \int_{b_-}^{b_+} u \circ f_b(x)\, (1 + wh(x)b)\, \tau(b)\mathrm{d}b\,. \qquad (5.2)$$

Since $\mathsf{E}(B) = \int b\,\tau(b)\mathrm{d}b = 0$, one has $T1 = 1$ and $\|T\|_{\infty\to\infty} = 1$. The operator T is thus, formally, the transition operator of some Markov chain on the circle. But, for every $b \in [b_-, b_+]$ and every $x \in \mathbb{T}$, one has

$$\mathrm{e}^{wh(x)b} \;=\; (1 + wh(x)b) \cdot \mathrm{e}^{\mathcal{O}(w^2)}\,.$$

Therefore, for every $u \in \mathrm{L}^1(\mathbb{T}; \mathbb{R}_+)$, for every $n \in \mathbb{N}$ satisfying $w^2 n \leq 1$, and for almost every $x \in \mathbb{T}$, one has

$$T^n u(x) \;\sim\; \mathsf{E}\big(\mathrm{e}^{w\sum_{k=1}^n h(X_{k-1}^x)B_k}\, u(X_n^x)\big)\,. \qquad (5.3)$$

Let $y \in \mathbb{T}$. The proof of Proposition 5.1 rests on the fact that, when T^n acts on a function $u \in \mathrm{L}^1_{B(y,w^2)}(\mathbb{T})$, it can be well approximated by an operator $S_{y,n}$ which can be explicitly studied. In order to define $S_{y,n}$, let us first introduce the convolution operator T_y on $\mathrm{L}^p(\mathbb{T})$, $1 \leq p \leq \infty$, by setting

$$T_y u(x) := \int u \circ g_b(x, y)\, (1 + wh(y)b)\, \tau(b)\mathrm{d}b\,, \qquad (5.4)$$

where

$$g_b(x, y) \;:=\; x \,+\, \vartheta \,+\, \Phi(y, b) \;=\; x \,+\, w \,+\, w\phi(y)b \,+\, w^2\psi(y)b^2 \,+\, \mathcal{O}(w^3)\,, \qquad (5.5)$$

with Φ defined as in (3.13), and ϕ and ψ defined as in (3.15b) and (3.15c). Then, one sets $S_{y,0} := \mathrm{Id}$, and defines each $n \in \mathbb{N}$

$$S_{y,n} := T_{y-nw} \cdots T_{y-w}. \tag{5.6a}$$

$$R_y := T - T_y. \tag{5.6b}$$

The core of our approximation scheme is described by equation (5.32) below, but let us now describe it heuristically. Let $z \in \mathbb{T}$, and let $u \in \mathrm{L}^1_{B(z,w^2)}(\mathbb{T};\mathbb{R}_+)$. The support of $u \circ f_b$ should be centered at $z - w$, and so $g_b(\bullet, z-w)$ is likely to be the best approximation of f_b, among all the maps $g_b(\bullet, y)$ ($y \in \mathbb{T}$). Therefore, one can think of T_z as one of the best approximations of T among all the operators T_y ($y \in \mathbb{T}$). One writes

$$T^n u = T^{n-1} R_{z-w} u + T^{n-1} T_{z-w} u, \tag{5.7}$$

where R_{z-w} is defined by (5.6b). The first term in the right hand side of (5.7) can be bounded by means of our estimates on R_y ($y \in \mathbb{T}$), in Lemmas 5.3 or 5.4 below. One is thus left with the second term. From the definition (5.4) of T_y ($y \in \mathbb{T}$), the function $T_{z-w}u$ will be approximately centered at $z - w$. One now approximates T by T_{z-2w} and one obtains

$$T^{n-1} T_{z-w} u = T^{n-2} R_{z-2w} T_{z-w} u + T^{n-2} T_{z-2w} T_{z-w} u.$$

Again, one is left with the second term. But, continuing that way, one finally needs to handle the term $T T_{z-(n-1)w} \ldots T_{z-w} u$, and one arrives to

$$T T_{z-(n-1)w} \cdots T_z u = R_{z-nw} T_{z-(n-2)w} \cdots T_{z-w} u + T_{z-nw} \cdots T_{z-w} u, \tag{5.8}$$

By the definition (5.6a), one has $T_{z-nw} \cdots T_{z-w} u = S_{z,n} u$. So, this time, the second term in (5.8) can be bounded from above and below by some explicit estimates contained in Lemma 5.2 below. By means of Lemmas 5.3 and 5.4, one thus needs to show that the sum of the terms containing an operator of the form R_y ($y \in \mathbb{T}$) do not destroy the estimate on $S_{z,n}u$.

The rest of the section is organized as follows. In Lemma 5.2, one obtains some bounds on the functions $S_{y,n}u$ for $u \in \mathrm{L}^1_{B(y,w^2)}(\mathbb{T})$. The same bounds should be obtained for a Gaussian of variance nw^2 centered at y. The proof turns out to be a straightforward computation, since the operators T_y are diagonal in Fourier space. Next, Lemmas 5.3 and 5.4 give us bounds on R_y. Lemma 5.4 is actually not crucial, and needs only to be used when $n < 8$, since then the function $S_{y,n}u$ may not be smooth enough for Lemma 5.3 to be applied. Some easy results about the localization of the functions $T^n u$ and $S_{y,n}u$, for $u \in \mathrm{L}^1_{B(y,Cw)}(\mathbb{T})$, are then given in Lemma 5.5. Finally, the proof of Proposition 5.1 is given.

Let us notice that, in Lemma 5.2, and consequently in the proof of Proposition 5.1, one has to distinguish between the case where $y \sim 0$, and the case where y is away from 0. This comes from the lack of diffusivity of the chain X around 0 (see property (iii) of Corollary 3.4).

Lemma 5.2. *Let $\epsilon > 0$. There exists $K > 0$ such that, for every $n \in \mathbb{N}$ satisfying $8 \le n \le w^{-2}$, for every $y \in \mathbb{T} - B(0,\epsilon)$, and for every $u \in \mathrm{L}^1_{B(y,w^2)}(\mathbb{T};\mathbb{R}_+)$, one has $S_{y,n}u \in \mathrm{C}^2(\mathbb{T})$ and, for every $x \in \mathbb{T}$,*

$$|\partial_x^l S_{y,n} u(x)| \le \frac{K\|u\|_1}{(w\sqrt{n})^{(l+1)}}, \qquad l = 0,1,2, \tag{5.9a}$$

$$\left|\sin^k \pi(x + wn - y) \cdot \partial_x^k S_{y,n} u(x)\right| \le \frac{K\|u\|_1}{w\sqrt{n}}, \quad k = 1,2. \tag{5.9b}$$

Moreover, when ϵ is small enough, there exists $K'(\epsilon) > 0$, with $K'(\epsilon) \to \infty$ as $\epsilon \to 0$, such that, for every $n \in \mathbb{N}$ satisfying $\epsilon \le w^2 n \le 2\epsilon$, for every $x, y \in \mathbb{T}$, and for every $u \in \mathrm{L}^1_{B(y,w^2)}(\mathbb{T};\mathbb{R}_+)$,

$$|S_{y,n} u(x)| \ge K'(\epsilon)\|u\|_1 \quad \textit{when} \quad |x + nw - y|_{\mathbb{T}} \le 10\epsilon. \tag{5.10}$$

The proof is deferred to the Appendix A.3.

LEMMA 5.3. *There exists $K > 0$ such that, for every $u \in \mathrm{C}^2(\mathbb{T})$ and every $y \in \mathbb{T}$, one has*

$$\begin{aligned}\|R_y u\|_\infty \;\le\; Kw^2\Big\{ &\big\|\sin\pi(\,\bullet\, - y - w)\cdot u'\big\|_\infty + w\|u'\|_\infty \\ &+ \big\|\sin^2\pi(\,\bullet\, - y - w)\,\cdot\, u''\big\|_\infty + w\|u''\|_\infty\Big\}\,.\end{aligned} \tag{5.11}$$

PROOF. One takes some $u \in \mathrm{C}^2(\mathbb{T})$, and one fixes $x, y \in \mathbb{T}$. From the definitions (5.2) and (5.4), one has

$$\begin{aligned}R_y u(x) \;\equiv\; (T - T_y)u(x) \;&=\; \int (u \circ f_b(x) - u \circ g_b(x,y))\,(1 + wh(x)b)\,\tau(b)\mathrm{d}b \\ &\quad + w(h(x) - h(y)) \int u \circ g_b(x,y)\, b\,\tau(b)\mathrm{d}b \\ &=: A_1 \,+\, A_2\,.\end{aligned}$$

It is enough to bound $|A_1|$ and $|A_2|$ by the right hand side of (5.11).

Let us first bound $|A_1|$. By the mean value theorem, and the definitions (3.15) and (5.5) of f_b and g_b, one has

$$u \circ f_b(x) \,-\, u \circ g_b(x,y) \;=\; u'(x + w + \xi_1)\Big(w\big[\phi(x) - \phi(y)\big]b \,+\, w^2\big[\psi(x) - \psi(y)\big]b^2 \,+\, \mathcal{O}(w^3)\Big)\,,$$

where $\xi_1 \equiv \xi_1(b)$ is such that

$$|\xi_1| \;\le\; w|\phi(x) - \phi(y)| \,+\, \mathcal{O}(w^2)\,. \tag{5.12}$$

By the mean value theorem again, one has

$$u'(x + w + \xi_1) \;=\; u'(x + w) \,+\, u''(x + w + \xi_2)\,\xi_1\,,$$

where $\xi_2 \equiv \xi_2(b)$ is such that $|\xi_2| \le |\xi_1|$.

Therefore, setting $\tilde\tau(b) = (1 + wh(x)b)\,\tau(b)$, one can write A_1 as

$$\begin{aligned}A_1 \;=\; u'(x + w) &\int \Big(w\big[\phi(x) - \phi(y)\big]b \,+\, w^2\big[\psi(x) - \psi(y)\big]b^2 \,+\, \mathcal{O}(w^3)\Big)\tilde\tau(b)\mathrm{d}b \\ &+ \int u''(x + w + \xi_2(b))\,\xi_1(b)\,\big(w\big[\phi(x) - \phi(y)\big]b \,+\, \mathcal{O}(w^2)\big)\tilde\tau(b)\mathrm{d}b\,.\end{aligned}$$

One has

$$|\phi(x) - \phi(y)| \;\lesssim\; |\sin\pi(x - y)| \qquad \text{and} \qquad |\psi(x) - \psi(y)| \;\lesssim\; |\sin\pi(x - y)|\,.$$

So, taking into account the bound (5.12) and the fact that $\int b\tau(b)\mathrm{d}b = 0$, one gets

$$\begin{aligned}|A_1| \;\lesssim\; &w^2|u'(x + w)|\,|\sin\pi(x - y)| \,+\, w^3\|u'\|_\infty \\ &+ w^2\int |u''(x + w + \xi_2(b))|\,\sin^2\pi(x - y)\,\tilde\tau(b)\mathrm{d}b \,+\, w^3\|u''\|_\infty\,.\end{aligned} \tag{5.13}$$

But one has $\sin^2\pi(x - y) \le \sin^2\pi(x + \xi_2 - y) + \mathcal{O}(w)$. So, inserting this last bound in (5.13), one sees that $|A_1|$ is bounded by the right hand side of (5.11).

Let us then bound $|A_2|$. By the mean value theorem and the definition (5.5) of g_b, one writes

$$u \circ g_b(x,y) \;=\; u(x + w) \,+\, u'(x + w + \xi)\,\mathcal{O}(w)\,,$$

where $\xi \equiv \xi(b) = \mathcal{O}(w)$. Therefore, taking into account that $\int b\tau(b)\mathrm{d}b = 0$ and that $|h(x) - h(y)| \lesssim |\sin\pi(x-y)|$, one obtains

$$\begin{aligned} |A_2| &\lesssim w^2 \int |\sin\pi(x-y)| \cdot |u'(x+w+\xi(b))| \cdot |b|\tau(b)\mathrm{d}b \\ &\lesssim w^2 \big(\|\sin\pi(\mathrm{Id}-y-w)\cdot u'\|_\infty + w\|u'\|_\infty \big) . \end{aligned}$$

This finishes the proof. □

LEMMA 5.4. *Let $K, \epsilon > 0$. Let $y \in \mathbb{T}$ be such that $|y|_{\mathbb{T}} \geq \epsilon$. Then there exists $K' > 0$ such that, for every $u \in \mathrm{L}^1_{B(y,Kw)}(\mathbb{T})$, one has*

$$\|R_y u\|_1 \leq K' w \|u\|_1 . \tag{5.14}$$

Moreover $Tu \in \mathrm{L}^\infty(\mathbb{T})$, and one has

$$\|Tu\|_\infty \leq K' \, w^{-1} \, \|u\|_1 . \tag{5.15}$$

PROOF. The constants introduced in this proof may depend on K and ϵ. Let $u \in \mathrm{L}^1_{B(y,Kw)}(\mathbb{T})$. One writes

$$Tu(x) = \int_{B(y,Kw)} t(x,z)u(z)\mathrm{d}z \quad \text{and} \quad T_y u(x) = \int_{B(y,Kw)} t_y(x,z)u(z)\mathrm{d}z , \tag{5.16}$$

where the functions t and t_y are obtained by performing a change of variables in the definitions (5.2) and (5.4) of T and T_y. Setting $F_x(b) := f_b(x)$ and $G_x(b) := g_b(x,y)$, where f_b and g_b are defined in (3.15) and (5.5), one obtains

$$\begin{aligned} t(x,z) &= (1 + wh(x)F_x^{-1}(z))\,\tau(F_x^{-1}(z))\,\partial_z F_x^{-1}(z) , \\ t_y(x,z) &= (1 + wh(y)G_x^{-1}(z))\,\tau(G_x^{-1}(z))\,\partial_z G_x^{-1}(z) . \end{aligned} \tag{5.17}$$

Let $z \in B(y,Kw)$ be given. Let us see that $t(\bullet, z)$ and $t_y(\bullet, z)$ are well defined functions. The support of $t(\bullet, z)$ (respectively of $t_y(\bullet, z)$) is the support of $\tau \circ F^{-1}_{(\bullet)}(z)$ (resp. of $\tau \circ G^{-1}_{(\bullet)}(z)$). The support of $\tau \circ F^{-1}_{(\bullet)}(z)$ is made of all the x such that

$$b_- \leq F_x^{-1}(z) \leq b_+ \;\Leftrightarrow\; f_{b_-}(x) \leq z \leq f_{b_+}(x) \;\Leftrightarrow\; f_{b_+}^{-1}(z) \leq x \leq f_{b_-}^{-1}(z) .$$

One obtains a similar relation for the support of $\tau \circ G^{-1}_{(\bullet)}(z)$ and one gets therefore

$$\mathrm{supp}(t(\bullet, z)) , \; \mathrm{supp}(t_y(\bullet, z)) \;\subset\; B(z, Cw) \;\subset\; B(y, C'w).$$

The hypothesis $|y|_{\mathbb{T}} \geq \epsilon$ ensures that the maps F_x and G_x are invertible when $x \in B(y, C'w)$, and actually that

$$\partial_b F_x(b) \gtrsim w \quad \text{and} \quad \partial_b G_x(b) \gtrsim w . \tag{5.18}$$

This shows in particular that $t(\bullet, z)$ and $t_y(\bullet, z)$ are bounded functions.

Let us now show (5.14). Taking (5.17) into account, one has, from the definition (5.6b) of R_y,

$$\|R_y u\|_1 \leq \int_{B(y,Kw)} |u(z)|\mathrm{d}z \int_{B(y,C'w)} |t(x,z) - t_y(x,z)|\mathrm{d}x. \tag{5.19}$$

It is therefore enough to show that, for every $z \in B(y, Kw)$, one has

$$\int_{B(y,C'w)} |t(x,z) - t_y(x,z)| \mathrm{d}x \;=\; \mathcal{O}(w)\,. \tag{5.20}$$

Let us take some $z \in B(y, Kw)$ and some $x \in B(y, C'w)$. Since $b_- \leq F_x^{-1}(z), G_x^{-1}(z) \leq b_+$, since τ is bounded, and since (5.18) holds, one finds, starting from (5.17), that

$$|t(x,z) - t_y(x,z)| \;\lesssim\; |\partial_z F_x^{-1}(z) - \partial_z G_x^{-1}(z)| \,+ w^{-1}|\tau(F_x^{-1}(z)) - \tau(G_x^{-1}(z))| \,+\, C. \tag{5.21}$$

For every $b \in [b_-, b_+]$, one has $\partial_b F_x(b) = w\phi(x) + \mathcal{O}(w^2)$ and $\partial_b G_x(b) = w\phi(y) + \mathcal{O}(w^2)$. Therefore

$$\begin{aligned} |\partial_z F_x^{-1}(z) - \partial_z G_x^{-1}(z)| \;&\leq\; \Big|\frac{1}{w\phi(x) + \mathcal{O}(w^2)} - \frac{1}{w\phi(y) + \mathcal{O}(w^2)}\Big| \\ &\lesssim\; w^{-1}|\phi(y) - \phi(x) + \mathcal{O}(w)| \;\lesssim\; 1\,, \end{aligned} \tag{5.22}$$

since $|y - x| = \mathcal{O}(w)$. Inserting thus (5.22) in (5.21), and then (5.21) in (5.20), one finds

$$\begin{aligned} \int_{B(y,Cw)} |t(x,z) - t_y(x,z)| \mathrm{d}x \;&\lesssim\; w^{-1} \int_{B(y,Cw)} |\tau(F_x^{-1}(z)) - \tau(G_x^{-1}(z))| \mathrm{d}x \,+\, \mathcal{O}(w) \\ &=:\; w^{-1} I \,+\, \mathcal{O}(w)\,. \end{aligned} \tag{5.23}$$

It remains thus to show that $I = \mathcal{O}(w^2)$. For this, let us define

$$D_1 \;:=\; \{x \in \mathbb{T} : b_- \leq F_x^{-1}(z) \leq b_+\}\,, \quad \text{and} \quad D_2 \;:=\; \{x \in \mathbb{T} : b_- \leq G_x^{-1}(z) \leq b_+\}\,.$$

One writes

$$I \;=\; \int_{D_1 \cap D_2} (\dots) + \int_{(D_1 \cap D_2)^c} (\dots) \;=:\; I_1 \,+\, I_2\,.$$

First, when $x \in D_1 \cap D_2$, one uses the fact that $\tau \in \mathrm{C}^1([b_-, b_+])$, that

$$|F_x^{-1}(z) - G_x^{-1}(z)| \;=\; \Big|\frac{z - x - w}{w\phi(x)} - \frac{z - x - w}{w\phi(y)} + \mathcal{O}(w)\Big| \;=\; \mathcal{O}(w)\,,$$

since $|z - x - w| = \mathcal{O}(w)$ and $|\phi(y) - \phi(x)| = \mathcal{O}(w)$, and that $\mathrm{Leb}(D_1 \cap D_2) = \mathcal{O}(w)$, to conclude that $I_1 = \mathcal{O}(w^2)$. Next, when $x \in (D_1 \cap D_2)^c$, one has $t(x,z) = t_y(x,z) = 0$, except on $D_1 \,\Delta\, D_2$. But, for every $b \in [b_-, b_+]$, one has $|f_b(x) - g_b(x,y)| = \mathcal{O}(w^2)$, since $|x - y| = \mathcal{O}(w)$. So, one has $\mathrm{Leb}(D_1 \,\Delta\, D_2) = \mathcal{O}(w^2)$, and thus $I_2 = \mathcal{O}(w^2)$.

Let us finally show (5.15). From (5.16), one has that $|Tu(x)| \leq \sup_{z \in B(y,Kw)} |t(x,z)|$. The relations (5.17) and (5.18) allow us to obtain the result. □

In order to prove the next lemma, we introduce the adjoint T^* of T with respect to the Lebesgue measure. This operator is defined on $\mathrm{L}^p(\mathbb{T})$ ($1 \leq p \leq \infty$) and is such that, for every $u \in \mathrm{L}^p(\mathbb{T})$ and every $v \in \mathrm{L}^{p'}(\mathbb{T})$, with $1/p + 1/p' = 1$, one has

$$\int_{\mathbb{T}} v\, T^* u \, \mathrm{d}x = \int_{\mathbb{T}} u\, Tv \, \mathrm{d}x\,. \tag{5.24}$$

From the definition (5.2) of T, one concludes that

$$T^* u(x) \;=\; \int_{b_-}^{b_+} u \circ f_b^{-1}(x) \left[1 + w\, h \circ f_b^{-1}(x) b\right] \partial_x f_b^{-1}(x)\, \tau(b) \mathrm{d}b\,. \tag{5.25}$$

Therefore, when $u \geq 0$, one has

$$T^* u(x) \;\geq\; \mathrm{e}^{-\mathcal{O}(w)} \int u \circ f_b^{-1}(x)\, \tau(b) \mathrm{d}b\,. \tag{5.26}$$

For $z \in \mathbb{R}$, let us define the chain $Y = (Y_n^z : n \in \mathbb{N}_0)$ by $Y_0^z := z$ and

$$Y_n^z \;:=\; f_{B_n}^{-1}(Y_{n-1}^z) \;=\; Y_{n-1}^z - \, w \,-\, w\phi(Y_{n-1}^z) B_n \,+\, \mathcal{O}(w^2)\,. \tag{5.27}$$

LEMMA 5.5. *Let $K > 0$. There exist $K_2 \geq K_1 > 0$ such that, for every $n \in \mathbb{N}$, for every $y \in \mathbb{T}$, and for every $u \in \mathrm{L}^1_{B(y,Kw)}(\mathbb{T})$, one has*

$$\mathrm{supp}(T^n u),\; \mathrm{supp}(S_{y,n} u) \;\subset\; \big[y - K_2 w n, y - K_1 w n\big]\,. \tag{5.28}$$

Morover, for every $R > 0$ large enough, there exists $K' > 0$ such that, for every $n \in \mathbb{N}$ satisfying $wn \leq 1$, for every $y \in \mathbb{T}$, and for every $u \in \mathrm{L}^1_{B(y,w)}(\mathbb{T}; \mathbb{R}_+)$, one has

$$\int_{B(y-nw, R\sqrt{w})} T^n u(z) \mathrm{d}z \;\geq\; K' \|u\|_1\,. \tag{5.29}$$

PROOF. Let us first show (5.28). Let us consider the case of $T^n u$; the case of $S_{y,n} u$ is strictly analogous. From the definition (5.2), one sees that

$$\mathrm{supp}(T^n u) \;\subset\; \big[f_{b_+}^{-n}(y - Kw/2), f_{b_-}^{-n}(y + Kw/2)\big]\,.$$

This implies the result, since, by the definition (3.15) of f_b, one has, for every $x \in \mathbb{T}$ and every $b \in [b_-, b_+]$,

$$(1 + b_-)w - \mathcal{O}(w^2) \;\leq\; x - f_b^{-1}(x) \;\leq\; (1 + b_+)w + \mathcal{O}(w^2)\,.$$

Let us then show (5.29). Let $u \in \mathrm{L}^1_{B(y,w^2)}(\mathbb{T}; \mathbb{R}_+)$, let $R > 0$, and let $n \in \mathbb{N}$ be such that $nw \leq 1$. From the definition (5.24) of the adjoint T^*, one has

$$\int_{B(y-nw, R\sqrt{w})} T^n u(z) \mathrm{d}z \;=\; \int_{B(y,w)} T^{*n} \chi_{B(y-nw, R\sqrt{w})}(z)\, u(z) \mathrm{d}z\,.$$

It is therefore enough to show that, for every $z \in B(y, w)$, one has $T^{*n} \chi_{B(y-nw,R\sqrt{w})}(z) \gtrsim 1$, if R is large enough. But, since $wn \leq 1$, (5.26) implies that

$$\begin{aligned} T^{*n} \chi_{B(y-nw,R\sqrt{w})}(z) \;&\gtrsim\; \mathsf{E}\big(\chi_{B(y-nw,R\sqrt{w})} \circ f_{B_n}^{-1} \circ \cdots \circ f_{B_1}^{-1}(z)\big) \\ &=\; 1 \,-\, \mathsf{P}\big(|Y_n^z - (y - nw)| \geq R\sqrt{w}\big)\,, \end{aligned} \tag{5.30}$$

where Y is defined in (5.27). Therefore, since $|z - y| = \mathcal{O}(w)$ and since $w^2 n = \mathcal{O}(w)$, one obtains, from the definition (5.27) of Y, and from Azuma's inequality (4.6), that

$$\mathsf{P}\big(|Y_n^z - (y - nw)| \geq R\sqrt{w}\big) \;=\; \mathsf{P}\Big(\Big|w \sum_{k=1}^{n} \phi(Y_{k-1}^z) + \mathcal{O}(w)\Big| \geq R\sqrt{w}\Big) \;\leq\; 2\mathrm{e}^{-\frac{CR^2}{nw}}\,. \tag{5.31}$$

The proof is finished by taking R large enough, and inserting (5.31) in (5.30). $\square$

PROOF OF PROPOSITION 5.1. Let $n \geq 9$ be such that $nw^2 \leq 1$. Let us make three observations. First, by (5.3), it is enough to show the proposition with $\mathsf{E}_x(\mathrm{e}^{w\sum_{k=1}^n h(X_{k-1})B_k}\, u(X_n))$ replaced by $T^n u(x)$ in (5.1a) and (5.1b).

Second, it is enough to prove the proposition for functions in $\mathrm{L}^1_{B(y,w^2)}(\mathbb{T};\mathbb{R}_+)$ for every $y \in \mathbb{T}$. So, throughout the proof, one assumes that $y \in \mathbb{T}$ is given, and the symbol v denotes a function in $\mathrm{L}^1_{B(y,w^2)}(\mathbb{T};\mathbb{R}_+)$.

Third, it is enough to show (5.1b) for some n' satisfying $w^2 n' \le 1/2$. Indeed, let us now assume that (5.1b) is shown for this n', and let n be such that $1/2 \le w^2 n \le 1$. From the definition (5.2), one sees that, if $u_1 \ge u_2$, one has $Tu_1 \ge Tu_2$. So, one writes $n = n' + n''$ and, for every $u \in \mathrm{L}^1(\mathbb{T},\mathbb{R}_+)$, one gets $T^n u(x) = T^{n''}T^{n'}u \gtrsim \|u\|_1 T^{n''}1 \sim \|u\|_1$, where the fact that $T^{n''}1 \sim 1$ directly follows from the definition (5.2) of T, Azuma's bound (4.4), and the hypothesis $w^2 n \le 1$.

The proof is now divided into three steps, but the core is entirely contained in the first one.

Step 1: approximating T^n by $S_{y,n}$: One here shows the bounds (5.1a) and (5.1b) under two particular assumptions:

1. One supposes that $|y|_{\mathbb{T}} \ge \epsilon_1$, for some $\epsilon_1 > 0$. The constants introduced below may depend on ϵ_1.
2. Only for (5.1b), one assumes that n is such that $\epsilon_2 \le n \le 2\epsilon_2$ and that $|x+nw-y|_{\mathbb{T}} \le 10\epsilon_2$ for some $\epsilon_2 > 0$ small enough.

By the definition (5.6a) of $S_{y,n}$, one can write

$$\begin{aligned} T^n v \;&=\; S_{y,n}v \,+\, \sum_{k=1}^{8} T^{n-k} R_{y-kw} S_{y,k-1} v \,+\, \sum_{k=9}^{n-1} T^{n-k} R_{y-kw} S_{y,k-1} v \\ &=:\; S_{y,n}v \,+\, Q_1 \,+\, Q_2 \,. \end{aligned} \tag{5.32}$$

Let us bound $\|Q_1\|_\infty$. Let $k \in \mathbb{N}$ be such that $1 \le k \le 8$. By (5.28), one has

$$\operatorname{supp}(R_{y-kw}S_{y,k-1}v) \;\subset\; B(y, Cw)\,. \tag{5.33}$$

Remembering that $\|T\|_{\infty\to\infty} = 1$, one uses (5.14) and (5.15) to obtain that

$$\begin{aligned} \|Q_1\|_\infty &\le \sum_{k=1}^{8} \|T^{9-k} R_{y-kw} S_{y,k-1} v\|_\infty \;\lesssim\; w^{-1} \sum_{k=1}^{8} \|R_{y-kw} S_{y,k-1} v\|_1 \\ &\lesssim \sum_{k=0}^{7} \|S_{y,k-1} v\|_1 \;\lesssim\; \|v\|_1 \,, \end{aligned} \tag{5.34}$$

where, for the last inequality, one has used the fact that $\|T_y\|_{1\to 1} = 1$ for every $y \in \mathbb{T}$.

Let us bound $\|Q_2\|_\infty$. By Lemma 5.3 and estimates (5.9b) and (5.9a) in Lemma 5.2, one has, for $8 \le k \le w^{-2}$,

$$\|T^{n-k} R_{y-kw} S_{y,k-1} v\|_\infty \;\le\; \|R_{y-kw} S_{y,k-1} v\|_\infty \;\lesssim\; w^2 \left\{ \frac{1}{w\sqrt{k}} + \frac{w}{w^2 k} + \frac{1}{w\sqrt{k}} + \frac{w}{w^3 k^{3/2}} \right\} \cdot \|v\|_1 .$$

Therefore, since $w^2 n \le 1$ by hypothesis, one gets

$$\|Q_2\|_\infty \;\lesssim\; (w\sqrt{n} + w \log n + C)\|v\|_1 \;\lesssim\; \|v\|_1 \,. \tag{5.35}$$

So, from (5.32), (5.34) and (5.35), one has

$$\|T^n v - S_{y,n} v\|_\infty \;\le\; C\,\|v\|_1 \,,$$

where the constant C is independent of ϵ_2. Therefore, in the particular case considered, (5.1a) follows from (5.9a) with $l = 0$, and (5.1b) follows from (5.10), if ϵ_2 has been chosen small enough.

Step 2: proof of (5.1a)**:** By Step 1, (5.1a) is known to hold when $|y|_{\mathbb{T}} \geq \epsilon_1$, and one may now assume that $|y|_{\mathbb{T}} < \epsilon_1$. Moreover, one has still the freedom to take ϵ_1 as small as we want. One now uses the hypothesis $nw \geq \kappa$. Let $m \in \mathbb{N}$ be such that $mw = \epsilon'$, for some $\epsilon' \in]0, c/2]$. If ϵ_1 is small enough, it follows from (5.28) that one can chose ϵ' such that $\mathrm{supp}(T^m u) \cap B(0, \epsilon_1) = \emptyset$. But the particular case considered in Step 1 implies that (5.1a) is valid for any function in $\mathrm{L}^1_{\mathbb{T}-B(0,\epsilon_1)}(\mathbb{T})$, and thus one has

$$T^n v(x) \;=\; T^{n-m} T^m v(x) \;\lesssim\; \frac{\|T^m v\|_1}{w\sqrt{n-m}} \;\lesssim\; \frac{\|v\|_1}{w\sqrt{n}}\,,$$

where the last inequality follows from the fact that $\|T\|_{1\to 1} \leq \mathrm{e}^{\mathcal{O}(w)}$, as can be seen from the definition (5.2).

Step 3: proof of (5.1b)**:** One first will establish (5.1b) for n such that $n = \lfloor \epsilon_2 w^{-2} \rfloor$, and for x such that $|x + nw - y|_{\mathbb{T}} \leq 10\epsilon_2$. By Step 1, it is now enough to consider the case $|y|_{\mathbb{T}} < \epsilon_1$. Let now $m = \lfloor \frac{1}{2} w^{-1} \rfloor$, and let $R > 0$. If R is taken large enough, it follows from (5.29), and from the particular case of (5.1b) already established in Step 1, that

$$T^n v(x) \;\geq\; T^{n-m}\big(\chi_{B(y-mw,R\sqrt{w})} T^m v\big)(x) \;\gtrsim\; \int_{B(y-mw,R\sqrt{w})} T^m v(z)\mathrm{d}z \;\gtrsim\; \|v\|_1\,.$$

One finally needs to get rid of the assumption $|x + nw - y|_{\mathbb{T}} \leq 10\epsilon_2$. One uses a classical technique [7]. One shows (5.1b) for $n = kq$, with $k \geq 1/18\epsilon_2$, and q such that $q = \lfloor \epsilon_2 w^{-2} \rfloor$. One already knows that

$$T^q v \;\gtrsim\; \chi_{B(y-qw,10\epsilon_2)} \|v\|_1\,. \tag{5.36}$$

But one now will show that, for every $z \in \mathbb{T}$, and for every $s \in [\epsilon_2, 1]$, one has

$$T^q \chi_{B(z,s)} \gtrsim \epsilon_2\, \chi_{B(z-qw,s+9\epsilon_2)}. \tag{5.37}$$

This will imply the result :

$$\begin{aligned} T^n v \;&=\; T^{kq} v \;\gtrsim\; T^{(k-1)q} \chi_{B(y-qw,10\epsilon_2)}\, \|v\|_1 \\ &\gtrsim\; \cdots \;\gtrsim\; \epsilon_2^{k-1} \chi_{B(y-kqw,(10+9(k-1))\epsilon_2)}\, \|v\|_1 \;\gtrsim\; \epsilon_2^{k-1} \|v\|_1\,. \end{aligned}$$

Let us thus show (5.37). Let $z \in \mathbb{T}$ and $s \in [\epsilon_2, 1]$. Let us write $T^q u(x) = \int t_q(x, z') u(z') \mathrm{d}z'$ for any $u \in \mathrm{L}^1(\mathbb{T})$. Relation (5.36) implies in fact that $t_q(x, \bullet\,) \gtrsim \chi_{B(x+qw,10\epsilon_2)}(\,\bullet\,)$ (which may be formally checked by taking $u(x) = \delta(y - x)$). Therefore

$$\begin{aligned} T^q \chi_{B(z,s)}(x) \;&\gtrsim\; \int \chi_{B(x+qw,10\epsilon_2)}(z') \cdot \chi_{B(z,s)}(z') \mathrm{d}z' \\ &\gtrsim\; \epsilon_2\, \chi_{B(z,s+9\epsilon_2)}(x + qw) \;=\; \epsilon_2\, \chi_{B(z-qw,s+9\epsilon_2)}(x)\,. \end{aligned}$$

This finishes the proof. ☐

6 Putting everything together

In [5] p. 1710, Casher and Lebowitz derive the lower bound $\mathsf{E}(J_n) \gtrsim (T_1 - T_n) n^{-3/2}$. However, their argument contains a gap, and consequently this lower bound remains still to be proven.

Indeed, their proof is based on the estimate on the following estimate of $D_n(e_1)$ ($K_{1,n}$ in their notation):

$$\mathsf{E}\big[D_n(e_1)^2\big] \;\sim\; \mathrm{e}^{Cnw^2} \quad \text{as} \quad w \searrow 0\,. \tag{6.1}$$

This bound is obtained by computing the eigenvalues of a 4×4 matrix F, defined in [5] p. 1710. But this estimate cannot hold. Indeed, we know for example, from Corollary 3.6 and Proposition 5.1, that $\mathsf{E}(D_{1,n}^2) \sim w^{-2}$ when $w^2 n \sim 1$. Although the computation of the eigenvalues of F is correct, the authors do not take into account the fact that a w-dependent change of variables is needed to obtain a correct estimate on $\mathsf{E}[D_n(e_1)^2]$.

6.1 Proof of the lower bound

We begin by a lemma. Let (L_n) and (K_n) be the processes defined in Lemma 3.7.

LEMMA 6.1. *For every $\alpha > 0$, there exists $C(\alpha) > 0$, such that, for every $a > 0$, and every $n \in \mathbb{N}$ satisfying $w^2 n \leq 1$, one has*

$$\mathsf{P}(|K_n| \geq a),\, \mathsf{P}(|L_n| \geq a) \;\leq\; C(\alpha) w^{\alpha}\,. \tag{6.2}$$

PROOF. Let $(A_n : n \in \mathbb{N}_0)$ be a $\mathbb{F}$-adapted process such that

$$A_n \;:=\; \mathrm{e}^{M_n + L_n + \mathcal{O}(w^2 n)} \tag{6.3}$$

for every $n \in \mathbb{N}_0$, with M_n as defined in Lemma 3.7. From the expressions (3.28) and (3.29), both K_n and L_n are of the form

$$R_n \;:=\; w^2 \sum_{j=1}^{n} A_{j-1} S_{j-1} B_j\,,$$

where (S_j) is $\mathbb{F}$-adapted, and satisfies $|S_j| \lesssim 1$ for $j \in \mathbb{N}_0$.

Let $a > 0$. One writes

$$\begin{aligned}\mathsf{P}(R_n \geq a) \;=\; &\mathsf{P}\Big(w^{3/2} \sum_{j=1}^{n} w^{1/2} A_{j-1} S_{j-1} B_j \geq a,\; \max_{1\leq j\leq n} w^{1/2} A_{j-1} \leq 1\Big)\\ &+ \mathsf{P}\Big(w^{3/2} \sum_{j=1}^{n} w^{1/2} A_{j-1} S_{j-1} B_j \geq a,\; \max_{1\leq j\leq n} w^{1/2} A_{j-1} > 1\Big)\,.\end{aligned}$$

Let us now define a process $(\tilde{A}_n : n \in \mathbb{N}_0)$ by setting $\tilde{A}_n := A_n \cdot \chi_{[0,1]}(w^{1/2} A_n)$. One has

$$\mathsf{P}(R_n \geq a) \;\leq\; \mathsf{P}\Big(w^{3/2} \sum_{j=1}^{n} w^{1/2} \tilde{A}_{j-1} S_{j-1} B_j \geq a\Big) \;+\; \sum_{j=1}^{n} \mathsf{P}(w^{1/2} A_{j-1} > 1)\,. \tag{6.4}$$

First, by Azuma's inequality (4.6), and since $w^2 n \leq 1$, one has

$$\mathsf{P}\Big(w^{3/2} \sum_{j=1}^{n} w^{1/2} \tilde{A}_{j-1} S_{j-1} B_j \geq a\Big) \;\leq\; 2\,\mathrm{e}^{-Ca^2/w^3 n} \;\leq\; \mathrm{e}^{-Ca^2 w^{-1}}\,. \tag{6.5}$$

Next, it follows from (3.27), (3.28) and (3.30) that A_n defined in (6.3) if also of the form $A_n = \mathrm{e}^{w \sum_{j=1}^{n} G_{j-1} B_j + \mathcal{O}(w^2 n)}$, where (G_j) is $\mathbb{F}$-adapted, and $|G_j| \lesssim 1$ for $j \in \mathbb{N}_0$. So, applying again Azuma's inequality, one gets

$$\mathsf{P}(w^{1/2} A_{j-1} > 1) \;=\; \mathsf{P}\Big(w \sum_{k}^{j-1} G_{k-1} B_k + \mathcal{O}(w^2 n) > \frac{1}{2} \log \frac{1}{w}\Big) \;\lesssim\; \mathrm{e}^{-\frac{C \log^2(1/w)}{(j-1) w^2}} \;\leq\; \mathrm{e}^{-C' \log^2(1/w)}\,.$$

Therefore $\sum_{j=1}^{n} \mathsf{P}(w^{1/2} A_{j-1} > 1) \lesssim w^{-2} \mathrm{e}^{-C \log^2(w^{-1})}$. The proof is finished by inserting this last bound and (6.5) in (6.4). □

With the help of this lemma we can now prove the lower bound $\mathsf{E} J_n^{\mathrm{CL}} \gtrsim n^{-3/2}$ of Theorem 1.1. Indeed, from (2.6), it follows that

$$\mathsf{E} J_n^{\mathrm{CL}} \gtrsim \int_{(2n)^{-1/2}}^{n^{-1/2}} \mathsf{E} j_n(w) \mathrm{d} w \,,$$

with j_n defined in (2.7). It is therefore enough to show that when $1/2 \leq w^2 n \leq 1$ the bound $\mathsf{E} j_n(w) \gtrsim w^2 \sim n^{-1}$ holds. So let $1/2 \leq w^2 n \leq 1$, and use Corollary 3.6 in (2.7) to write

$$j_n(w) \gtrsim \left\{ 1 + \frac{(\Gamma_n^\vartheta \sin X_n^\vartheta)^2}{w^4} + \frac{(\Gamma_{n-1}^\vartheta \sin X_{n-1}^\vartheta)^2}{w^2} + \frac{(\Gamma_n^0 \sin X_n^0)^2}{w^2} + (\Gamma_{n-1}^0 \sin X_{n-1}^0)^2 \right\}^{-1} . \tag{6.6}$$

Let us take some $R, c > 1$. The constants introduced below may depend on R and c. Let us observe that, by point (i) of Corollary 3.4, one has $|X_{n-1}|_{\mathbb{T}} \lesssim w$ provided $|X_n|_{\mathbb{T}} \lesssim w^2$, and that, from the definition (3.17), one has $\Gamma_{n-1} \in [0, 2R]$ when $\Gamma_n \in [0, R]$. It follows therefore from (6.6) that

$$\mathsf{E} j_n(w) \gtrsim \mathsf{P}\big(|X_n^\vartheta|_{\mathbb{T}} \leq w^2, \, \Gamma_n^\vartheta \leq R, \, |X_n^0|_{\mathbb{T}} \leq c w, \, \Gamma_n^0 \leq c R \big) . \tag{6.7}$$

We now uses Lemma 3.7. First, by (3.25), one has

$$\chi_{B(cw)}(X_n^0) \geq \chi_{[0,R]}(\mathrm{e}^{M_n}) \cdot \chi_{B(0,1)}(L_n) \cdot \chi_{B(0,w^2)}(X_n^\vartheta) \,, \tag{6.8}$$

provided c is large enough. Secondly, by (3.26), one has

$$\chi_{[0,cR]}(\Gamma_n^0) \geq \chi_{B(0,1)}(K_n) \cdot \chi_{[0,R]}(\Gamma_n^\vartheta) \,, \tag{6.9}$$

again, provided c is large enough. Using then (6.8) and (6.9) in (6.7), one obtains

$$\begin{aligned}
\mathsf{E} j_n(w) &\gtrsim \mathsf{P}(|X_n^\vartheta|_{\mathbb{T}} \leq w^2, \, \Gamma_n^\vartheta \leq R, \, \mathrm{e}^{M_n} \leq R, \, |L_n| \leq 1, \, |K_n| \leq 1) \\
&\geq \mathsf{P}(|X_n^\vartheta|_{\mathbb{T}} \leq w^2, \, |L_n| \leq 1, \, |K_n| \leq 1) \\
&\quad - \mathsf{P}(|X_n^\vartheta|_{\mathbb{T}} \leq w^2, \, \Gamma_n^\vartheta > R) - \mathsf{P}(|X_n^\vartheta|_{\mathbb{T}} \leq w^2, \, \mathrm{e}^{M_n} > R) \\
&\geq \mathsf{P}(|X_n^\vartheta|_{\mathbb{T}} \leq w^2) - \mathsf{P}(|L_n| > 1) - \mathsf{P}(|K_n| > 1) \\
&\quad - \mathsf{P}(|X_n^\vartheta|_{\mathbb{T}} \leq w^2, \, \Gamma_n^\vartheta > R) - \mathsf{P}(|X_n^\vartheta|_{\mathbb{T}} \leq w^2, \, \mathrm{e}^{M_n} > R) .
\end{aligned}$$

Applying then Markov's inequality to the two last terms, one gets

$$\begin{aligned}
\mathsf{E} j_n(w) &\gtrsim \mathsf{P}(|X_n^\vartheta|_{\mathbb{T}} \leq w^2) - \mathsf{P}(|L_n| > 1) - \mathsf{P}(|K_n| > 1) \\
&\quad - \frac{1}{R} \mathsf{E}\Big[\chi_{B(0,w^2)}(X_n^\vartheta) \cdot \Gamma_n^\vartheta \Big] - \frac{1}{R} \mathsf{E}\Big[\chi_{B(0,w^2)}(X_n^\vartheta) \cdot \mathrm{e}^{M_n} \Big] .
\end{aligned}$$

Proposition 5.1 and Lemma 6.1 allow then to conclude that $\mathsf{E}(j_n(w)) \gtrsim w^2$ if R is chosen large enough. This finishes the proof.

6.2 Proof of the upper bound

Let $n \in \mathbb{N}$. Let $c > 0$ to be fixed later. Starting from (2.6), one writes

$$\mathsf{E} J_n^{\mathrm{CL}} \sim \int_0^{c/n} \mathsf{E} j_n(w) \mathrm{d}w + \int_{c/n}^{w_0} \mathsf{E} j_n(w) \mathrm{d}w + \int_{w_0}^{\infty} \mathsf{E} j_n(w) \mathrm{d}w =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 , \tag{6.10}$$

with j_n defined in (2.7). Using the crude bounds $D^2_{n-1}(e_1), D^2_n(e_2), D^2_{n-1}(e_2) \geq 0$ in the definition of j_n, and applying then Corollary 3.6, one obtains

$$j_n(w) \lesssim \frac{1}{1 + w^{-2} D_n^2(e_1)} \lesssim h(\Gamma_n^\vartheta \sin \pi X_n^\vartheta) \quad \text{with} \quad h(r) = \frac{1}{1 + w^{-4} r^2} . \tag{6.11}$$

Let us first bound J_1. Let $w \in [0, c/n[$. First, $\Gamma_n^\vartheta \gtrsim 1$, as can be checked from its definition (3.17). Next, if c is small enough, one has, by point (i) of Corollary 3.4, that

$$wn \lesssim X_n \leq \frac{1}{2} wn \leq \frac{1}{2} .$$

Therefore one has $\sin^2 \pi X_n^\vartheta \gtrsim w^2 n^2$, and thus

$$\mathcal{J}_1 \lesssim \int_0^{c/n} \frac{\mathrm{d}w}{1 + w^{-2} n^2} \lesssim n^{-3} . \tag{6.12}$$

Let us next bound $\mathcal{J}_2$. Let $w \in [c/n, w_0[$, and $m = \min\{n, \lfloor w^{-2} \rfloor\}$. One writes

$$\mathsf{E} j_n(w) = \int_{\mathbb{R}} \int_{\mathbb{T}} \mathsf{E}\big(j_n(w) | X_{n-m}^\vartheta = x, \Gamma_{n-m}^\vartheta = a\big)\, \mathsf{P}(X_{n-m}^\vartheta \in \mathrm{d}x, \Gamma_{n-m}^\vartheta \in \mathrm{d}a) . \tag{6.13}$$

To simplify notations, set $\mathsf{E}(\bullet | x, a) := \mathsf{E}(\bullet | X_{n-m}^\vartheta = x, \Gamma_{n-m}^\vartheta = a)$. If $x \in \mathbb{T}$ and $a \in \mathbb{R}$ are given, it follows from (6.11) that

$$\mathsf{E}(j_n(w) | x, a) \lesssim \mathsf{E} h(a \Gamma_m^x \sin \pi X_m^x) , \tag{6.14}$$

since, by the definition (3.17), one may write $\Gamma_n^\vartheta = \prod_{l=1}^n g(X_{l-1}^\vartheta, B_l) = \Gamma_{n-m}^\vartheta \prod_{l=n-m+1}^n g(X_{l-1}^\vartheta, B_l)$, for some function g. Because $h(r) \leq 1$ and $h(r) \leq w^4 r^{-2}$ for every $r \in \mathbb{R}$, one has, for every event A, the bound

$$h(a\, \Gamma_m^x \sin \pi X_m^x) \leq 1_A + 1_{A^c} \cdot w^4 \cdot (a\, \Gamma_m^x \sin \pi X_m^x)^{-2} . \tag{6.15}$$

So, taking $1_A = \chi_{[0,1]}(w^{-4} a^2 \sin^2 \pi X_m^x)$, and using (6.15) in (6.14) one obtains

$$\mathsf{E}(j_n(w) | x, a) \lesssim \mathsf{E}\Big\{ \chi_{[0,1]}(w^{-4} a^2 \sin^2 \pi X_m^x) + \chi_{]1,\infty[}(w^{-4} a^2 \sin^2 \pi X_m^x) \cdot w^4 \cdot (a\, \Gamma_m^x \sin \pi X_m^x)^{-2} \Big\} .$$

Therefore, Proposition 5.1 implies

$$\begin{aligned}
\mathsf{E}(j_n(w) | x, a) &\lesssim \frac{1}{w\sqrt{m}} \int_{\mathbb{T}} \{ \chi_{[0,1]}(w^{-4} a \sin^2 \pi y) + \chi_{]1,\infty[}(w^{-4} a \sin^2 \pi y)\, w^4 a^{-2} \sin^{-2} \pi y \} \mathrm{d}y \\
&\lesssim \frac{1}{w\sqrt{m}} \int_{\mathbb{T}} \frac{\mathrm{d}y}{1 + w^{-4} a^2 \sin^2 \pi y} \lesssim \frac{1}{w\sqrt{m}} \int_{-1/2}^{1/2} \frac{\mathrm{d}y}{1 + (w^{-2} a y)^2} \\
&\leq \frac{w^2 a^{-1}}{w\sqrt{m}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{1 + z^2} \lesssim \frac{w}{\sqrt{m}} a^{-1} ,
\end{aligned}$$

where one has used the change of variables $z = w^{-2}ay$ to get the third line. One now inserts this last bound in (6.13). Applying Proposition 4.1, one gets

$$\begin{aligned}\mathsf{E}j_n(w) &\lesssim \int_{\mathbb{R}}\int_{\mathbb{T}} \frac{w}{\sqrt{m}}\, a^{-1}\, \mathsf{P}(X^{\vartheta}_{n-m} \in \mathrm{d}x,\, \Gamma^{\vartheta}_{n-m} \in \mathrm{d}a) \\ &= \frac{w}{\sqrt{m}}\mathsf{E}(1/\Gamma^{\vartheta}_{n-m}) \lesssim \frac{w}{\sqrt{m}}\mathrm{e}^{-\alpha w^2(n-m)} \lesssim \max\Big\{\frac{w}{\sqrt{n}}, w^2\Big\}\,\mathrm{e}^{-\alpha w^2 n}.\end{aligned}$$

Therefore

$$\mathcal{J}_2 \lesssim \frac{1}{\sqrt{n}}\int_0^{n^{-1/2}} w\mathrm{e}^{-\alpha w^2 n}\mathrm{d}w + \int_{n^{-1/2}}^{\infty} w^2\mathrm{e}^{-\alpha w^2 n}\mathrm{d}w \lesssim n^{-3/2}\,. \tag{6.16}$$

It has already been shown by O'Connor [17] that $\mathcal{J}_3 \lesssim \mathrm{e}^{-Cn^{1/2}}$. One thus finishes the proof by inserting this last estimate, together with (6.12) and (6.16) in (6.11).

6.3 On other heat baths

Associate a heat bath to a function $\mu : \mathbb{R} \to \mathbb{C}$ as described by Dhar [8]. One may then obtain, at least formally, a new heat bath by replacing μ with a function $\tilde{\mu} : \mathbb{R} \to \mathbb{C}$ defined by scaling $\tilde{\mu}(w) \sim \mu(\mathrm{sgn}(w)|w|^s)$, $s > 0$. In [8] Dhar argued based on numerics and a non-rigorous approximation that Casher-Lebowitz and Rubin-Greer bath functions $\mu_{\mathrm{CL}}(w) \sim iw$ and $\mu_{\mathrm{RG}}(w) \sim \mathrm{e}^{-\mathrm{i}\pi\vartheta(w)}$, with $\vartheta(w)$ given in (3.6), yield $\mathsf{E}J_n^{\widetilde{\mathrm{CL}}} \sim n^{-(1+s/2)}$ and $\mathsf{E}J_n^{\widetilde{\mathrm{RG}}} \sim n^{-(1+|s-1|)/2}$, respectively. The first of these statements can be proven rigorously by directly adapting the proof of Theorem 1.1. The second case, however, does not follow directly from the proof of $\mathsf{E}J^{\mathrm{RG}} \sim n^{-1/2}$, even though we believe it should not be too difficult to prove by using our results.

To see where the difficulties within this second case lie, as well as to further demonstrate our approach, let us sketch how $\mathsf{E}J^{\mathrm{RG}} \sim n^{-1/2}$, first proven by Verheggen [23], can be obtained by using our representation of $D_n(v)$. Indeed, the choices $\tilde{e}_1 := 2^{-1/2}(e_1 + e_2)$ and $\tilde{e}_2 := 2^{-1/2}(e_1 - e_2)$ yield (Proposition 3.5) $D_n(\tilde{e}_1) \sim \Gamma_n^{x_1}\sin\pi X_n^{x_1}$ and $D_n(\tilde{e}_2) \sim w^{-1}\Gamma_n^{x_2}\sin\pi X_n^{x_2}$ with $x_1 = 1/2 + \mathcal{O}(w)$ and $x_2 = w/2 + \mathcal{O}(w^2)$, respectively. If one substitutes these in the expression for the current density $j_n^{\mathrm{RG}}(w)$ of the Rubin-Greer model (the equation between 3.1 and 3.2 in [23]) one ends up with an estimate

$$(1 + (\Gamma_n^{x_1})^2 + (\Gamma_n^{x_2})^2)^{-1} \lesssim j_n^{\mathrm{RG}}(w) \lesssim (1 + (\Gamma_n^{x_2})^2)^{-1}\,, \quad \text{for} \quad w \le w_0\,, \tag{6.17}$$

after making use of the basic properties of X-processes (Corollary 3.4). This reveals that the Rubin-Greer model is special in the sense that the random phases $X_n^{x_k}$ in the expressions $D_n(\tilde{e}_k) \sim \Gamma_n^{x_k}\sin\pi X_n^{x_k}$ do not have any direct role in the scaling behavior of the current. The reason why proving $\mathsf{E}J_n^{\widetilde{\mathrm{RG}}} \sim n^{-(1+|s-1|)/2}$, $s \neq 1$, is again more difficult is that the bounds analogous to (6.17) become again explicitly depended on X^{x_k}.

Now continuing with the RG-model, based on (6.17) one can prove $\mathsf{E}j_n^{\mathrm{RG}}(w) \sim \mathrm{e}^{-Cw^2 n}$ which then implies the scaling: $\mathsf{E}J_n^{\mathrm{RG}} = \int_{\mathbb{R}}\mathsf{E}j_n^{\mathrm{RG}}(w)\mathrm{d}w \sim n^{-1/2}$. Indeed, for the lower bound $\mathsf{E}j_n^{\mathrm{RG}}(w) \gtrsim \mathrm{e}^{-Cw^2 n}$ one considers the typical behavior, which is easier to analyze than in the Casher-Lebowitz model since X-processes are not present. The respective upper bound follows from Proposition 4.1.

A Appendix

A.1 Proof of Lemma 3.2

By using (3.1) one gets

$$f_b \;\equiv\; g^{-1}\circ\mathcal{M}_A\circ g \;=\; E\circ\mathcal{M}_{G^{-1}AG}\circ E^{-1}\,,$$

where

$$G^{-1}AG \;=\; \begin{bmatrix}(1+\mathrm{i}\delta)\mathrm{e}^{\mathrm{i}\pi\vartheta} & -\mathrm{i}\delta\mathrm{e}^{\mathrm{i}\pi\vartheta}\\ \mathrm{i}\delta\mathrm{e}^{-\mathrm{i}\pi\vartheta} & (1-\mathrm{i}\delta)\mathrm{e}^{-\mathrm{i}\pi\vartheta}\end{bmatrix}, \tag{A.1}$$

and

$$\delta \;=\; \frac{\pi^2w^2b}{2\sin\pi\vartheta} \;=\; \frac{(\pi w/2)b}{\sqrt{1-(\pi w/2)^2}} \;=\; (\pi w/2)\,b \;+\; \mathcal{O}(w^3b)\,. \tag{A.2}$$

Here the second equality follows from (3.6).

The map $\mathcal{M}_{G^{-1}AG}$ describes the evolution $\xi\mapsto\mathcal{M}_A(\xi)$ on the complex unit circle ∂D:

$$\mathcal{M}_{G^{-1}AG}(\mathrm{e}^{\mathrm{i}\phi}) \;=\; \mathrm{e}^{\mathrm{i}(\phi+2\pi\vartheta)}\,\frac{1\;+\;\mathrm{i}\delta\,(1-\mathrm{e}^{-\mathrm{i}\phi})}{1\;-\;\mathrm{i}\delta\,(1-\mathrm{e}^{\mathrm{i}\phi})} \;=:\; \exp\bigl[\mathrm{i}(\phi+2\pi\vartheta+2\tilde{\Phi}(\phi,\delta))\bigr]\,.$$

Here the effect of noise δ comes through

$$\begin{aligned}\tilde{\Phi}(\phi,\delta) \;&=\; \arg\bigl[1+\mathrm{i}\delta(1-\mathrm{e}^{-\mathrm{i}\phi})\bigr] \;=\; \arctan\left[\frac{1-\cos\phi}{1-\delta\sin\phi}\,\delta\right]\\ &=\; (1-\cos\phi)\,\delta \;+\; (1-\cos\phi)\sin\phi\,\delta^2 \;+\; \mathcal{O}\bigl((1-\cos\phi)\delta^3\bigr)\,.\end{aligned} \tag{A.3}$$

By substituting $\phi=2\pi x$ and using the middle expression of (A.2) in place of δ we obtain (3.13a).

Let $h(w,x,b)$ be a function so that $wb\,h(w,x,b)\sin^2\!\pi x$ equals the argument of arctan in (3.13a). It is easy to see that h is a smooth bounded function on $[0,w_0]\times\mathbb{T}\times[b_-,b_+]$. We may then write $\Phi(x,b)\equiv\Phi(w;x,b)$ as

$$\Phi(w,x,b) \;=\; \frac{1}{\pi}wb\,h(w,x,b)\sin^2\!\pi x \;+\; \frac{1}{6\pi}\arctan'''(s)\bigl[wb\,h(w,x,b)\sin^2\!\pi x\bigr]^3\,, \tag{A.4}$$

where the third derivative $\arctan'''(s)$ of arctan is bounded on $0\le s\le wb\sin^2(\pi x)h(w,x,b)=\mathcal{O}(w)$. By expanding $h(w,x,b)=\pi+\mathcal{O}(w)$ similarly, and then substituting the result back into (A.4) one obtains (3.13b).

To prove the formula (3.12b) for f_b^{-1} we note that $\mathrm{e}^{\mathrm{i}2\pi f_b^{-1}(y)}=\mathcal{M}_{\widetilde{\Lambda}}^{-1}(\mathrm{e}^{\mathrm{i}2\pi y})=\mathcal{M}_{\widetilde{\Lambda}^{-1}}(\mathrm{e}^{\mathrm{i}2\pi y})$ where $\widetilde{\Lambda}$ is the matrix in (A.1). After replacing $\widetilde{\Lambda}$ by its inverse, the proof proceeds just like before. The identity involving $\Phi(x,-b)$ follows by expressing f_b and f_b^{-1} in terms of Φ in $x=f_b^{-1}(f_b(x))$.

A.2 Proof of Lemma 4.2

Both proofs are rather directly adapted from Freedman's paper [11]. We start with Freedman's bound. To this end define a function $g:\mathbb{R}\to\mathbb{R}$: $g(0)=1/2$, $g(t):=(\mathrm{e}^t-1-t)/t^2$ for $t\neq0$. Let $t,y\in\mathbb{R}$ so that $|y|\le1$. By definition we have then

$$\mathrm{e}^{ty} \;=\; 1\;+\;ty\;+\;(ty)^2g(ty)\,.$$

It is not too difficult to see that g is an increasing function. Therefore, $g(ty)\le g(t)$ above, and

$$\mathrm{e}^{ty} \;\le\; 1\;+\;ty\;+\;y^2t^2g(t) \;=\; 1\;+\;ty\;+\;y^2(\mathrm{e}^t-1-t) \;\equiv\; 1\;+\;ty\;+\;y^2\kappa_1(t)\,. \tag{A.5}$$

Suppose Y is a random variable such that $|Y| \le 1$ and $\mathsf{E}(Y) = 0$. Setting $y = Y$ in (A.5) and taking expectation yields

$$\mathsf{E}\,\mathrm{e}^{tY} \;\le\; \mathsf{E}\big(1 \,+\, tY \,+\, \kappa_1(t)Y^2\big) \;=\; 1 \,+\, \kappa_1(t)\mathsf{E}(Y^2) \;\le\; \mathrm{e}^{\kappa_1(t)\mathsf{E}(Y^2)}\,. \tag{A.6}$$

Now, set $Y_i := (M_i - M_{i-1})/m$, so that $|Y_i| \le 1$ and $\mathsf{E}(Y_i|\mathcal{F}_{i-1}) = 0$. By using $\kappa_m(t) = m^{-2}k_1(tm)$ to write $\kappa_m(t)(M_i - M_{i-1})^2 = \kappa_1(mt)Y_i^2$, the estimate (A.6) implies that for any $t \in \mathbb{R}$:

$$\mathsf{E}\Big(\mathrm{e}^{t(M_i-M_{i-1}) \,-\, \kappa_m(t)\mathsf{E}[(M_i-M_{i-1})^2|\mathcal{F}_{i-1}]}\Big|\mathcal{F}_{i-1}\Big) \;=\; \mathsf{E}\Big(\mathrm{e}^{tmY_i \,-\, \kappa_1(tm)\mathsf{E}[Y_i^2|\mathcal{F}_{i-1}]}\Big|\mathcal{F}_{i-1}\Big) \;\le\; 1\,. \tag{A.7}$$

Recall the definition (4.3) of V_n and the pointwise bound $V_n \le v_n$. Apply these to get the first two lines below. Then use (A.7) iteratively to get Freedman's bound:

$$\begin{aligned}
\mathsf{E}\,\mathrm{e}^{tM_n} \;&\le\; \mathrm{e}^{\kappa_m(t)v_n}\,\mathsf{E}\,\mathrm{e}^{tM_n - \frac{1}{2}\kappa_m(t)V_n} \\
&=\; \mathrm{e}^{\kappa_m(t)v_n}\,\mathsf{E}\Big\{\mathrm{e}^{tM_{n-1} - \frac{1}{2}\kappa_m(t)V_{n-1}}\,\mathsf{E}\Big(\mathrm{e}^{t(M_n-M_{n-1}) \,-\, \kappa_m(t)\mathsf{E}[(M_n-M_{n-1})^2|\mathcal{F}_{i-1}]}\Big|\mathcal{F}_{i-1}\Big)\Big\} \\
&\le\; \mathrm{e}^{\kappa_m(t)v_n}\,\mathsf{E}\,\mathrm{e}^{tM_{n-1} - \frac{1}{2}\kappa_m(t)V_{n-1}} \;\le\; \cdots \;\le\; \mathrm{e}^{\kappa_m(t)v_n}\,.
\end{aligned}$$

The bound (4.5) comes from the power expansion $k_m(t) = (1/2)t^2 + k_m'''(s)t^3 = (1/2)t^2 + (m/6)\mathrm{e}^{ms}t^3$, with $s \in [0, t]$, by taking $s = |t|$.

The proof of Azuma's bound proceeds in a very similar way: First, one uses the convexity of the exponent function to get a bound

$$\mathrm{e}^{ty} \;=\; \mathrm{e}^{\frac{1+y}{2}t + \frac{1-y}{2}(-t)} \;\le\; \frac{1+y}{2}\mathrm{e}^{t} \,+\, \frac{1-y}{2}\mathrm{e}^{-t} \;=\; \cosh t \,+\, y\sinh t \;\le\; \mathrm{e}^{t^2/2} + y\sinh t\,,$$

for every $t, y \in \mathbb{R}$ with $|y| \le 1$. Using this instead of (A.5) in the first inequality of (A.6) yields the bound $\mathsf{E}\,\mathrm{e}^{tY} \le \mathrm{e}^{\frac{1}{2}t^2}$, and consequently $\mathsf{E}\big(\mathrm{e}^{t(M_i-M_{i-1})}\big|\mathcal{F}_{i-1}\big) = \mathsf{E}\big(\mathrm{e}^{(tm)Y_i}\big|\mathcal{F}_{i-1}\big) \le \mathrm{e}^{(tm)^2/2}$. Iterating this finishes the proof:

$$\mathsf{E}\mathrm{e}^{tM_n} \;=\; \mathsf{E}\big\{\mathrm{e}^{tM_{n-1}}\mathsf{E}(\mathrm{e}^{t(M_i-M_{i-1})}|\mathcal{F}_{n-1})\big\} \;=\; \mathrm{e}^{t^2m^2/2}\mathsf{E}(\mathrm{e}^{tM_{n-1}}) \;\le\; \cdots \;\le\; \mathrm{e}^{t^2m^2n/2}\,.$$

A.3 Proof of Lemma 5.2

Let us start with some conventions and definitions: For $k \in \mathbb{N}_0$, $y \in \mathbb{T}$ we define:

$$y_k := y - kw, \quad \alpha_k := \phi(y_k), \quad \gamma_k := h(y_k)\,.$$

For $\epsilon > 0$ and $n_0 \in \mathbb{N}$, one defines

$$H(\epsilon, n_0) \;:=\; \big\{(y, n) \in \mathbb{T} \times \mathbb{N} : |y|_{\mathbb{T}} \ge \epsilon, n \ge n_0,\, w^2 n \ge \epsilon\big\}\,.$$

For $u \in \mathrm{L}^1(\mathbb{T})$ and $\xi \in \mathbb{Z}$, one defines

$$\hat{u}(\xi) = \int_{\mathbb{T}} u(x)\mathrm{e}^{-\mathrm{i}2\pi\xi x}\mathrm{d}x\,.$$

The operators T_{y_k} ($k \in \mathbb{N}$) are diagonal in Fourier space: for every $\xi \in \mathbb{Z}$, one has

$$\widehat{(T_{y_k}u)}(\xi) \;=\; \mathrm{e}^{\mathrm{i}2\pi w\xi}\lambda_k(w\xi) \cdot \hat{u}(\xi)\,, \tag{A.8}$$

where λ_k is a function on $\mathbb{R}$ defined by

$$\begin{aligned}
\lambda_k(z) \;:&=\; \int \mathrm{e}^{\mathrm{i}2\pi z w^{-1}(\vartheta - w + \Phi(y_k, b))}(1 + w\gamma_k b)\tau(b)\mathrm{d}b \\
&=\; \int \mathrm{e}^{\mathrm{i}2\pi z(\alpha_k b + \mathcal{O}(w))}(1 + w\gamma_k b)\tau(b)\mathrm{d}b.
\end{aligned} \tag{A.9}$$

Let $y \in \mathbb{T}$, let $u \in \mathrm{L}^1_{B(y,w^2)}(\mathbb{T};\mathbb{R}_+)$, and let $v \in \mathrm{L}^1_{B(0,w^2)}(\mathbb{T};\mathbb{R}_+)$ be such that

$$v(x) = u(x+y). \tag{A.10}$$

One writes

$$S_{y,n}u(x) \;=\; T_{y_n}\cdots T_{y_1}v(x-y) \;=\; \sum_{\xi\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi\xi(x+nw-y)}\Lambda_n(\xi)\hat{v}(\xi). \tag{A.11}$$

where Λ_n is a function on $\mathbb{R}$ defined by

$$\Lambda_n(\xi) := \prod_{j=1}^{n} \lambda_j(\xi w) \quad (n \geq 1). \tag{A.12}$$

But, if $(y,n) \in H(\epsilon, 8)$ for some $\epsilon > 0$, the right hand side of (A.11) represents actually a C^2-function. This follows directly from (A.17) with $l = 0$ in Lemma A.1 below, and the fact that $|\hat{v}(\xi)| \leq \|v\|_1$ for every $\xi \in \mathbb{Z}$.

Lemma A.1. *Let $\epsilon > 0$. There exist $K, K', \epsilon' > 0$ such that, for every $(y,n) \in H(\epsilon,1)$, and for every $\xi \in \mathbb{R}$ satisfying $|\xi w| \leq \epsilon'$, one has*

$$\mathrm{e}^{-Knw^2\xi^2} \leq |\Lambda_n(\xi)| \leq \mathrm{e}^{-K'nw^2\xi^2}, \tag{A.13}$$

$$|\Lambda_n'(\xi)| \leq Knw^2(1+|\xi|)\mathrm{e}^{-K'nw^2\xi^2}, \tag{A.14}$$

$$|\Lambda_n''(\xi)| \leq Knw^2(1+nw^2+nw^2\xi^2)\mathrm{e}^{-K'nw^2\xi^2}, \tag{A.15}$$

$$|\arg(\Lambda_n(\xi))| \leq Knw^2(|\xi| + w|\xi|^3). \tag{A.16}$$

For every $\epsilon' > 0$, there exist $K, K' > 0$ such that, for every $(y,n) \in H(\epsilon,1)$, and for every $\xi \in \mathbb{Z}$ satisfying $|\xi w| > \epsilon'$, one has

$$|\partial_\xi^l \Lambda_n(\xi)| \;\leq\; \frac{K(wn)^l}{(1+K'|\xi w|)^{n/2}}, \quad l = 0,1,2. \tag{A.17}$$

Proof. The constants introduced in this proof may depend on ϵ. For the whole proof, one sets $z = \xi w$. Before starting, let us make two observations. First, one has $|\alpha_k| \lesssim 1$ and $|\gamma_k| \lesssim 1$ for every $k \in \mathbb{N}_0$. Secondly, for every $(y,n) \in H(\epsilon,1)$, there exists an integer $m \geq n/2$ independent of y, and a subsequence

$$\{k_j\} \;\equiv\; \{k_j : 1 \leq j \leq m\} \;\subset\; \{1,2,\ldots,n\} \tag{A.18}$$

such that $|\alpha_{k_j}| \gtrsim 1$.

Let us first prove the formulas (A.13) up to (A.16). One takes $(y,n) \in H(\epsilon,1)$. A Taylor expansion in (A.9), taking into account that $\int \tau(b)\mathrm{d}b = 1$ and $\int b\tau(b)\mathrm{d}b = \mathsf{E}(B) = 0$, gives

$$\lambda_k(z) \;=\; 1 + \mathrm{i}\mathcal{O}(w|z|) \;-\; \frac{(2\pi)^2}{2}z^2\alpha_k^2\mathsf{E}(B^2) \;+\; \mathcal{O}(wz^2) \;+\; \mathrm{i}\mathcal{O}(|z|^3) \;+\; \mathcal{O}(z^4)\,,$$

as $z \to 0$. Therefore, one has

$$|\lambda_k(z)| \;=\; \mathrm{e}^{-\frac{(2\pi)^2}{2}z^2\alpha_k^2\mathsf{E}(B^2) + \mathcal{O}(z^2w+|z|^3)}\,, \tag{A.19}$$

$$|\arg(\lambda_k(z))| \;=\; \mathcal{O}(|z|w + |z|^3)\,, \tag{A.20}$$

as $z \to 0$. Similarly, a Taylor expansion in (A.9) gives

$$|\partial_z \lambda_k(z)| \;=\; \mathcal{O}(w+|z|)\,, \tag{A.21}$$

$$|\partial_z^2 \lambda_k(z)| \;=\; \mathcal{O}(1)\,, \tag{A.22}$$

as $z \to 0$.

First, by (A.19) with $z = \xi w$, and by the definition (A.12) of Λ_n, one obtains

$$|\Lambda_n(\xi)| = \exp\Big[-\frac{1}{2}(\xi w)^2 \mathsf{E}(B^2)\sum_{k=1}^{n}\alpha_k^2 + \mathcal{O}\big(n(\xi w)^2(|\xi w|+w)\big)\Big] \quad \text{as} \quad \xi w \to 0\,.$$

This shows (A.13), taking into account the two observations at the beginning of this proof. Next, with $z = \xi w$, one has

$$\partial_\xi \Lambda_n(\xi) \;=\; w\sum_{j=1}^{n} \partial_z \lambda_j(z) \prod_{\substack{1\le k\le n\\ k\ne j}} \lambda_k(z)\,, \tag{A.23}$$

$$\partial_\xi^2 \Lambda_n(\xi) \;=\; w^2\sum_{j=1}^{n}\Big(\partial_z^2 \lambda_j(z) \prod_{\substack{1\le k\le n\\ k\ne j}} \lambda_k(z) + \partial_z\lambda_j(z)\sum_{\substack{1\le k\le n\\ k\ne j}}\partial_z\lambda_k(z)\prod_{\substack{1\le l\le n\\ l\ne j,k}}\lambda_l(z)\Big)\,. \tag{A.24}$$

One then obtains (A.14) and (A.15), by using these last formulas together with (A.21), (A.22), and the fact that $|\lambda_k(z)| \le 1$ for every $k \in \mathbb{N}$ and every $z \in \mathbb{R}$, which follows from the definition (A.9). Finally, (A.16) directly follows from (A.20).

Let us now show (A.17). Let $\epsilon' > 0$, and let $(y,n) \in H(\epsilon,1)$. The constants introduced below may depend on ϵ'. One proceeds in two steps.

First, one shows (A.17) for $|z| = |\xi w| \in [\epsilon', 1/\epsilon'[$. It is actually enough to show that

$$|\lambda_k(z)| \le 1-\epsilon_1 \tag{A.25}$$

for some $\epsilon_1 > 0$ and for every $k \in \{k_j\}$, with $\{k_j\}$ as defined in (A.18). Indeed, from the definition (A.9), one has $|\lambda_k(z)| \le 1$ and $|\partial_z^l \lambda_k(z)| \lesssim 1$ for $l = 1,2$, for every $k \in \mathbb{N}$ and every $z \in \mathbb{R}$. So, inserting (A.25) in (A.12), (A.23) or (A.24), respectively for $l=0$, $l=1$ or $l=2$, will imply

$$|\partial_\xi^l \Lambda_n(\xi)| \;\lesssim\; (wn)^l(1-\epsilon_1)^{\frac{n}{2}-2}\,,$$

which is equivalent to (A.17) when $\epsilon' \le |\xi w| < 1/\epsilon'$.

So let us show (A.25). By continuity of τ, one finds an interval J on which $\tau \ge \epsilon_2$ for some $\epsilon_2 > 0$. One has

$$\int_J \mathrm{e}^{\mathrm{i}z(\alpha_k b+\mathcal{O}(w))}(1+w\gamma_k b)\tau(b)\mathrm{d}b = \int_J \mathrm{e}^{\mathrm{i}z\alpha_k b}\tau(b)\mathrm{d}b + \mathcal{O}(w)\,,$$

and, for some $\epsilon_3 > 0$,

$$|\int_J \mathrm{e}^{\mathrm{i}z\alpha_k b}\tau(b)\mathrm{d}b| \le (1-\epsilon_3)\int_J \tau(b)\mathrm{d}b\,.$$

Therefore

$$|\lambda_k(z)| \le (1-\epsilon_3)\int_J \tau(b)\mathrm{d}b + \int_{J^{\mathrm{c}}}\tau(b)\mathrm{d}b + \mathcal{O}(w) = 1-\epsilon_3\int_J\tau(b)\mathrm{d}b + \mathcal{O}(w) \le 1-\epsilon_2\epsilon_3\mathrm{Leb}(J) + \mathcal{O}(w)\,.$$

One thus may take $\epsilon_1 = \frac{1}{2}\epsilon_2\epsilon_3\mathrm{Leb}(J)$.

Next, one shows (A.17) for $|z| = |\xi w| \geq 1/\epsilon'$. Here, it is enough to show that, for some $C > 0$, one has

$$|\partial_z^l \lambda_k(z)| \leq C/|z|\,, \tag{A.26}$$

for $l = 0, 1, 2$, and for every $k \in \{k_j\}$. Indeed, if ϵ' has been taken small enough, one finds some $C' > 0$ such that $C/|z| \leq 1/(1 + C'|z|)$, when $|z| \geq 1/\epsilon'$. So, inserting now (A.26) in (A.12), (A.23) or (A.24), respectively for $l = 0$, $l = 1$ or $l = 2$, one will obtain (A.17) for $|\xi w| \geq 1/\epsilon'$.

So let us show (A.26). It follows from (A.9) that $\partial_z^l \lambda_k(z)$ can be written under the form $\partial_z^l \lambda_k(z) = \int \mathrm{e}^{\mathrm{i}z\mu(b)} \rho_l(b) \mathrm{d}b$. An integration by parts gives

$$\partial_z^l \lambda_k(z) \;=\; \frac{1}{z} \frac{\mathrm{e}^{\mathrm{i}z\mu(b)\rho(b)}}{\mathrm{i}\partial_b \mu(b)} \Big|_{b_-}^{b_+} - \frac{1}{z} \int_{b_-}^{b_+} \mathrm{e}^{\mathrm{i}z\mu(b)} \partial_b \Big(\frac{\rho(b)}{\mathrm{i}\partial_b \mu(b)} \Big) \mathrm{d}b\,.$$

Here, one has $\rho_l \in \mathrm{C}^1([b_-, b_+])$, since $\tau \in \mathrm{C}^1([b_-, b_+])$, and $|\partial_b^j \rho(b)| \lesssim 1$ for $j = 0, 1$. Moreover, one checks form the definition (3.13) of Φ that $|\partial_b \mu(b)| \gtrsim 1$ and $|\partial_b^2 \mu(b)| \lesssim 1$. This finishes the proof. □

One now let $(y, n) \in H(\epsilon, 8)$. The constants introduced below depend on y only through ϵ.

Proof of (5.9a). By (A.11), (A.13) and (A.17) (with $l = 0$), there exists $\epsilon' > 0$ such that,

$$\begin{aligned} |\partial_x^l S_{y,n} u(x)| \;&\leq\; (2\pi)^2 \sum_{\xi \in \mathbb{Z}} |\xi|^l |\Lambda_n(\xi)| \, \|u\|_1 \\ &\lesssim\; \|u\|_1 \sum_{\xi : |\xi w| \leq \epsilon'} |\xi|^l \mathrm{e}^{-Cn(\xi w)^2} + \|u\|_1 \sum_{\xi : |\xi w| > \epsilon'} \frac{|\xi|^l}{(1 + C|\xi w|)^{\frac{n}{2}}} \\ &\lesssim\; \frac{\|u\|_1}{w^{l+1}} \int_0^\infty y^l \mathrm{e}^{-Cny^2} \mathrm{d}y \;+\; \frac{\|u\|_1}{w^{l+1}} \int_{\epsilon'}^\infty \frac{y^l \mathrm{d}y}{(1 + Cy)^{\frac{n}{2}}} \\ &=: \; \frac{\|u\|_1}{w^{l+1}} (I_1 + I_2)\,. \end{aligned}$$

But one has $I_1 \lesssim n^{-(l+1)/2}$, and

$$I_2 \;\leq\; \frac{1}{C^l} \int_{\epsilon'}^\infty \frac{\mathrm{d}y}{(1 + Cy)^{\frac{n}{2} - l}} \;\leq\; \frac{1}{C^{l+1} (\frac{n}{2} - l - 1)(1 + C\epsilon')^{\frac{n}{2} - l - 1}} \;\lesssim\; \mathrm{e}^{-C(\epsilon')n}\,. \tag{A.27}$$

This finishes the proof. □

Proof of (5.9b). We will only consider the case $k = 2$; the case $k = 1$ can be handled similarly, and turns out to be easier. To simplify the notations, one writes

$$A \;:=\; \sin^2 \pi(x + nw - y) \cdot \partial_x^2 S_{y,n} u(x)\,.$$

We recall that the function v defined in (A.10) satisfies $v(x) = u(x + y)$. One has $\sin^2 z = \frac{1}{4}(2 - \mathrm{e}^{\mathrm{i}2z} - \mathrm{e}^{-\mathrm{i}2z})$, and thus, by (A.11), one has

$$\begin{aligned} A \;&=\; -\pi^2 \Big\{ 2 - \mathrm{e}^{\mathrm{i}2\pi(x+nw-y)} - \mathrm{e}^{-\mathrm{i}2\pi(x+nw-y)} \Big\} \sum_{\xi \in \mathbb{Z}} \xi^2 \mathrm{e}^{\mathrm{i}2\pi\xi(x+wn-y)} \Lambda_n(\xi) \hat{v}(\xi) \\ &=\; -\pi^2 \sum_{\xi \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi\xi(x+wn-y)} \Big\{ 2\xi^2 \Lambda_n(\xi) \hat{v}(\xi) \;-\; (\xi - 1)^2 \Lambda_n(\xi - 1) \hat{v}(\xi - 1) \\ &\qquad\qquad -\; (\xi + 1)^2 \Lambda_n(\xi + 1) \hat{v}(\xi + 1) \Big\}\,. \end{aligned} \tag{A.28}$$

Since

$$|\hat{v}(\xi) - \hat{v}(\xi-1)| \;\leq\; \int_{B(0,w^2)} |v(x)|\,|1-\mathrm{e}^{\mathrm{i}2\pi x}|\mathrm{d}x \;\lesssim\; w^2\|u\|_1\,,$$

for every $\xi \in \mathbb{Z}$, one has, for every $\epsilon' > 0$,

$$\begin{aligned}
|A| \;&\lesssim\; \|u\|_1 \sum_{\xi\in\mathbb{Z}} \Big|2\xi^2\Lambda_n(\xi) - (\xi-1)^2\Lambda_n(\xi-1) - (\xi+1)^2\Lambda_n(\xi+1)\Big| \;+\; (\xi w)^2|\Lambda_n(\xi)| \\
&\lesssim \|u\|_1 \sum_{\xi\in\mathbb{Z}} \Big\{\xi^2\big|2\Lambda_n(\xi) - \Lambda_n(\xi-1) - \Lambda_n(\xi+1)\big| \\
&\qquad\qquad + |\xi|\cdot\big|\Lambda_n(\xi-1) - \Lambda_n(\xi+1)\big| \;+\; (1+(\xi w)^2)|\Lambda_n(\xi)|\Big\} \\
&\lesssim\; \|u\|_1 \sum_{\xi\in\mathbb{Z}} \Big\{\xi^2|\Lambda_n''(\xi_1(\xi))| \;+\; |\xi|\cdot|\Lambda_n'(\xi_2(\xi))| \;+\; (1+(\xi w)^2)|\Lambda_n(\xi)|\Big\} \\
&=\; \|u\|_1 \sum_{\xi:|\xi w|\leq\epsilon'} (\cdots) \;+\; \sum_{\xi:|\xi w|>\epsilon'} (\cdots) \;=:\; \|u\|_1(I_1+I_2)\,.
\end{aligned} \tag{A.29}$$

The numbers $\xi_1(\xi)$ and $\xi_2(\xi)$ in (A.29) are obtained by a Taylor expansion and satisfy $|\xi_1(\xi)-\xi| \leq 2$ and $|\xi_2(\xi)-\xi| \leq 2$.

If ϵ' is taken small enough, then, by (A.13), (A.14) and (A.15), and because $nw^2 \leq 1$ by hypothesis, one has

$$I_1 \;\lesssim\; \int_0^\infty \big\{1 \,+\, (\xi w\sqrt{n})^2 \,+\, (\xi w\sqrt{n})^4\big\}\,\mathrm{e}^{-C(\sqrt{n}w\xi)^2}\mathrm{d}\xi \;\lesssim\; \frac{1}{w\sqrt{n}}\,. \tag{A.30}$$

By (A.17), one gets as for (A.27),

$$I_2 \lesssim \sum_{|\xi|w\geq\epsilon'} \frac{(\xi wn)^2+\xi wn+1}{(1+C\xi w)^{\frac{n}{2}}} \;\lesssim\; \frac{1}{w}\int_{\epsilon'}^\infty \frac{\big((yn)^2+yn+1\big)\mathrm{d}y}{(1+Cy)^{\frac{n}{2}}} \;\lesssim\; \frac{1}{w}\mathrm{e}^{-C'n}\,. \tag{A.31}$$

Inserting (A.30) and (A.31) in (A.29) gives the result. □

Proof of (5.10). Let $\epsilon > 0$ be as small as we want. One takes $x, y \in \mathbb{T}$ such that $|x+nw-y|_{\mathbb{T}} \leq 10\epsilon$. The constants introduced below do not depend on ϵ. We recall that the function v defined in (A.10) satisfies $v(x) = u(x+y)$. Starting from (A.11), one obtains

$$S_{y,n}u(x) \;\geq\; \sum_{\xi:|\xi|\leq\epsilon^{-2/3}} \mathrm{e}^{2\mathrm{i}\pi\xi(x+nw-y)}\Lambda_n(\xi)\hat{v}(\xi) \;-\; \sum_{\xi:|\xi|>\epsilon^{-2/3}} |\Lambda_n(\xi)|\,\|u\|_1\,. \tag{A.32}$$

On the one hand, mimicking the proof of (5.9a) with $l = 0$, and taking the hypothesis $nw^2 \geq \epsilon$ into account, one finds, for some $\epsilon' > 0$,

$$\begin{aligned}
\sum_{\xi:|\xi|>\epsilon^{-2/3}} |\Lambda_n(\xi)| \;&\lesssim\; \frac{1}{w}\int_{\epsilon^{-2/3}w}^{\epsilon'} \mathrm{e}^{-Cny^2}\mathrm{d}y + \frac{1}{w}\int_{\epsilon'}^\infty \frac{\mathrm{d}y}{(1+Cy)^{\frac{n}{2}}} \\
&\lesssim\; \frac{1}{\sqrt{\epsilon}}\int_{\epsilon^{-1/6}}^\infty \mathrm{e}^{-Cz^2}\mathrm{d}z + \frac{1}{\sqrt{\epsilon}}\mathrm{e}^{-C'(\epsilon')n} \;\lesssim\; \mathrm{e}^{-C''\epsilon^{-1/3}}\,,
\end{aligned} \tag{A.33}$$

where, to get rid of the term $\frac{1}{\sqrt{\epsilon}}\mathrm{e}^{-C'(\epsilon')n}$, one has used the hypothesis $nw^2 \geq \epsilon$, which implies $n \geq \epsilon^{-1/3}$ when w is small enough.

On the other hand, $\Lambda_n(-\xi) = \Lambda_n^*(\xi)$ by (A.12), $\hat{v}(-\xi) = \hat{v}^*(\xi)$ since u is real, and $\hat{v}(0) = \|u\|_1$ since $u \geq 0$. Therefore

$$\begin{aligned} &\sum_{\xi:|\xi|\leq\epsilon^{-2/3}} \mathrm{e}^{2\mathrm{i}\pi\xi(x+nw-y)}\Lambda_n(\xi)\hat{v}(\xi) \\ &\quad = \|u\|_1 + 2 \sum_{1\leq\xi\leq\epsilon^{-2/3}} |\Lambda_n(\xi)|\,|\hat{v}(\xi)| \cos\arg\left[\mathrm{e}^{2\mathrm{i}\pi\xi(x+nw-y)}\Lambda_n(\xi)\hat{v}(\xi)\right]. \end{aligned} \tag{A.34}$$

Since $v \in \mathrm{L}^1_{B(0,w^2)}(\mathbb{T})$, one has $\arg(\hat{v}(\xi)) \lesssim |\xi|w^2$ for every $\xi \in \mathbb{Z}$. So, by (A.16) and the hypothesis $|x + nw - y|_{\mathbb{T}} \leq 10\epsilon$, one obtains

$$\left|\arg\left(\mathrm{e}^{2\mathrm{i}\pi\xi(x+nw-y)}\Lambda_n(\xi)\hat{v}(\xi)\right)\right| \lesssim \epsilon\xi \lesssim \epsilon^{1/3}, \tag{A.35}$$

when $1 \leq \xi \leq \epsilon^{-2/3}$. But, if $1 \leq \xi \leq \epsilon^{-2/3}$, one has

$$|\hat{v}(\xi)| \geq \int v(x)\mathrm{d}x - \int_{B(0,w^2)} |v(x)|\,|\mathrm{e}^{-\mathrm{i}2\pi\xi x} - 1|\mathrm{d}x \geq (1 - C\xi w^2)\|u\|_1 \geq \frac{1}{2}\|u\|_1, \tag{A.36}$$

and, by (A.13), one has $|\Lambda_n(\xi)| \geq \mathrm{e}^{-Cnw^2\xi^2} \geq \mathrm{e}^{-C\epsilon\xi^2}$, since $nw^2 \geq \epsilon$. Therefore, using this last estimate, (A.35) and (A.36) in (A.34) gives

$$\sum_{\xi:|\xi|\leq\epsilon^{-2/3}} \mathrm{e}^{2\mathrm{i}\pi\xi(x+nw-y)}\Lambda_n(\xi)\hat{v}(\xi) \gtrsim \sum_{|\xi|\leq\epsilon^{-2/3}} \mathrm{e}^{-C'\epsilon\xi^2}\|v\|_1, \tag{A.37}$$

if ϵ is small enough.

Therefore, inserting (A.33) and (A.37) in (A.32), one gets

$$S_{y,n}u(x) \geq \|u\|_1\Big(C_1 \sum_{|\xi|\leq\epsilon^{-2/3}} \mathrm{e}^{-C'\epsilon\xi^2} - C_2\,\mathrm{e}^{-C''\epsilon^{-1/3}}\Big),$$

and this tends to ∞ as $\epsilon \to 0$. □

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